Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH4241-WE01

Title:
Representation Theory IV

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

SECTION A

1. (a) What is an intertwining map between two representations of a group? Carefully state Schur's Lemma for finite groups. **[4]**

(b) Let G be a finite group and let Z be its centre. Let (π, V) be an irreducible representation of G. Show that for every $z \in Z$ the action of $\pi(z)$ on V is a G-intertwiner. Conclude that for all $z \in Z$ there exists a nonzero scalar $\lambda(z) \in \mathbb{C}^\times$ such that

$$\pi(z)v = \lambda(z)v$$

for all $v \in V$. **[6]**

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2. (a) Give the character table of the symmetric group S_3. (No justification required, but explain all the notions used in the table.) **[4]**

(b) Consider $\mathbb{C}^2$ with its standard basis e_1, e_2 and let $W = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ (which has a standard basis $e_{ijk} := e_i \otimes e_j \otimes e_k$ with $1 \leq i, j, k \leq 2$). Let π be the representation of S_3 on W given by permuting the tensor factors on W. (For example, $\pi(12)(u \otimes v \otimes w) = v \otimes u \otimes w$, or $\pi(123)(u \otimes v \otimes w) = w \otimes u \otimes v$.) Compute the character of π and decompose π into irreducible representations. **[6]**

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3. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let Z be the centre of G.

(a) Define the centre $\mathfrak{z}$ of $\mathfrak{g}$. **[2]**

(b) Let $X \in \mathfrak{g}$. Show that, if $\exp(tX) \in Z$ for all $t \in \mathbb{R}$, then $X \in \mathfrak{z}$. Deduce that $\mathrm{Lie}(Z) \subset \mathfrak{z}$. *You may use the formula* $\exp(\mathrm{ad}_X) = \mathrm{Ad}_{\exp(X)}$. **[8]**

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4. (a) Decompose the $\mathfrak{sl}_{2,\mathbb{C}}$-representation $\mathrm{Sym}^2(\mathbb{C}^2) \otimes \mathbb{C}^2$ into irreducible representations, where $\mathbb{C}^2$ is the standard representation. **[5]**

(b) Find a weight basis for the irreducible subrepresentation of $\mathrm{Sym}^2(\mathbb{C}^2) \otimes \mathbb{C}^2$ of largest dimension. **[5]**

SECTION B

5. Let G be a finite group whose conjugacy classes are given by

Order	1	4	4	3
Conj. class	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$

Show that its character table is uniquely determined with this information by reconstructing it using the following steps.

(a) Find the dimensions of the irreducible representations of G. **[4]**

(b) Find the order of the commutator subgroup $C(G)$ and describe $C(G)$ as a union of conjugacy classes. **[3]**

(c) For each one-dimensional representation of G find (with justification) its character values. **[5]**

(d) Complete the table. **[3]**

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6. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of n and consider the associated Young permutation module $\mathcal{M}^\lambda$ of the symmetric group S_n. You may use that $\mathcal{M}^\lambda \simeq \mathrm{Ind}_{R(T)}^{S_n} 1$, the induced representation of the trivial representation of the row group $R(T) \simeq S_{\lambda_1} \times \cdots \times S_{\lambda_k}$ for any tabloid T of shape λ. (Yes, this does not depend on the actual filling.)

(a) Use Frobenius reciprocity to give a general formula for the multiplicity of a representation π of S_n occurring in $\mathcal{M}^\lambda$.

Apply this to $\pi = 1$, the trivial representation, and show that it occurs exactly once in $\mathcal{M}^\lambda$ for any partition λ of n. **[3]**

(b) Consider the standard representation $V = V_n$ of S_n. Describe the standard representation in terms of the permutation representation of S_n acting on $\{1, 2, \ldots, n\}$. Give a basis of V_n.

Use this to show that for $n \geq 3$ the restriction of V_n to the smaller symmetric group S_{n-2} acting on $\{1, 2, \ldots, n-2\}$ is given by

$$\mathrm{Res}_{S_{n-2}}^{S_n} V_n \simeq V_{n-2} \oplus 2 \cdot 1.$$

(So the trivial representation occurs with multiplicity two.) **[7]**

(c) Consider the partition $\lambda = (n-2, 1, 1)$ of $n \geq 4$. Describe the general form of Young's Rule for $\mathcal{M}^\lambda$. Which of the multiplicities can you determine using the previous parts? **[5]**

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7. (a) Let $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Let

$$G = \{g \in GL_2(\mathbb{R}) : g^T J g = J\}.$$

Use the exponential criterion to find the Lie algebra $\mathfrak{g}$ of G. **[5]**

(b) Find the number of connected components of G. **[5]**

(c) Now let $J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Let

$$G = \{g \in GL_3(\mathbb{R}) : g^T J g = J\}.$$

Write down (without proof) the Lie algebra $\mathfrak{g}$ of G. By finding a suitable basis, or otherwise, show that $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_{2,\mathbb{R}}$. **[5]**

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8. (a) State the definitions of a weight vector and a highest weight vector in a representation of $\mathfrak{sl}_{3,\mathbb{C}}$. **[3]**

(b) Let $\mathbb{C}^3$ be the standard representation of $\mathfrak{sl}_{3,\mathbb{C}}$ and let $V = \mathrm{Sym}^2(\mathbb{C}^3)$. Draw the weight diagram of V and write down a highest weight vector together with its weight. **[4]**

(c) Let $W = \Lambda^2(V)$. Find a dominant weight α such that the α-weight space $W_\alpha \subset W$ has dimension two. **[4]**

(d) Show that W_α contains no highest weight vectors. **[4]**
