Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH4261-WE01

Title:
Stochastic Analysis IV

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

SECTION A

1. Let $W = (W_t)_{t\in\mathbb{R}_+}$ be a Brownian motion with respect to a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, and let

$$X_t = W_t^2$$

for each $t \in \mathbb{R}_+$. In this question, you may use the fact that, if $Z \sim \mathcal{N}(0, \sigma^2)$, then $\mathbb{E}[Z^4] = 3\sigma^4$.

(a) Show that

$$\mathbb{E}\big[(X_t - X_s)^2\big] = 2(t^2 - s^2) + (t - s)^2$$

for any $0 \le s < t$. **[5]**

(b) For some $0 \le s < t$, let

$$\{s = t_0^n < t_1^n < \cdots < t_{k_n}^n = t\} \quad \text{for} \quad n \in \mathbb{N},$$

be a sequence of partitions of the interval $[s, t]$, such that the mesh size $\delta_n = \max_{1\le i\le k_n}(t_i^n - t_{i-1}^n)$ satisfies $\delta_n \to 0$ as $n \to \infty$. Further, for each $n \in \mathbb{N}$, let

$$\xi_n = \sum_{i=1}^{k_n} (X_{t_i^n} - X_{t_{i-1}^n})^2.$$

Find the value of $\lim_{n\to\infty} \mathbb{E}[\xi_n]$. **[5]**

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2. Let $X = (X_n)_{n\in\mathbb{Z}_+}$ be a stochastic process, adapted to a filtration $(\mathcal{F}_n)_{n\in\mathbb{Z}_+}$, such that $\mathbb{E}[|X_n|] < \infty$ for every $n \in \mathbb{Z}_+$. Let us assume that

$$X_n = X_0 + A_n + M_n$$

for every $n \in \mathbb{Z}_+$, where $A = (A_n)_{n\in\mathbb{Z}_+}$ is previsible, $M = (M_n)_{n\in\mathbb{Z}_+}$ is a martingale, and $A_0 = M_0 = 0$.

(a) State what it means for the stochastic process $A = (A_n)_{n\in\mathbb{Z}_+}$ to be previsible with respect to $(\mathcal{F}_n)_{n\in\mathbb{Z}_+}$. **[1]**

(b) Suppose that we can also write

$$X_n = X_0 + \widetilde{A}_n + \widetilde{M}_n$$

for every $n \in \mathbb{Z}_+$, where $\widetilde{A} = (\widetilde{A}_n)_{n\in\mathbb{Z}_+}$ is previsible, $\widetilde{M} = (\widetilde{M}_n)_{n\in\mathbb{Z}_+}$ is a martingale, and $\widetilde{A}_0 = \widetilde{M}_0 = 0$.
Prove that $A = \widetilde{A}$ and $M = \widetilde{M}$. **[5]**

(c) Prove that X is a submartingale if and only if A is an increasing process (in the sense that $A_{n+1} \ge A_n$ almost surely for every $n \in \mathbb{Z}_+$). **[4]**

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3. (a) Using Itô's formula, prove that if $M = (M_t)_{t \in \mathbb{R}_+}$ is a continuous local martingale, and $\lambda \in \mathbb{R}$ is a constant, then the process

$$\mathcal{E}^\lambda(M)_t = \exp\left(\lambda M_t - \frac{\lambda^2}{2}\langle M, M\rangle_t\right), \qquad t \in \mathbb{R}_+,$$

is a local martingale. **[4]**

(b) If B is a standard Brownian motion, use the result in part (a) to prove that $\mathcal{E}^\lambda(B)$ is a martingale. **[6]**

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4. (a) Let B be a Brownian motion on $\mathbb{R}$, and let K be a progressive process on the same filtered probability space, satisfying

$$\mathbb{E}\left[\int_0^t K_s^2 \,\mathrm{d}s\right] < \infty$$

for every $t > 0$.

Denote by H^2 the set of the L^2-bounded and continuous martingales (i.e., all continuous martingales $M = (M_t)_{t \in \mathbb{R}_+}$ satisfying $\sup_{t \in \mathbb{R}_+} \mathbb{E}[M_t^2] < \infty$), and H_0^2 the subset of H^2 vanishing at 0.

Use the stopping technique and stochastic integration theory on H_0^2 to construct the stochastic integral $\int_0^t K_s \,\mathrm{d}B_s$ for all $t > 0$. Also, state whether or not the process

$$\int_0^t K_s \,\mathrm{d}B_s, \quad t \in \mathbb{R}_+,$$

is in H_0^2, and explain why. **[5]**

(b) Let B be a Brownian motion on $\mathbb{R}$. Verify that you can use part (a) of this question to define the integral $M_t = \int_0^t \exp(B_s) \,\mathrm{d}B_s$, and calculate both $\langle M, M\rangle_t$ and $\mathbb{E}[M_t^2]$. **[5]**

SECTION B

5. Let $\mathbb{P}$ and $\mathbb{Q}$ be probability measures on a measurable space $(\Omega, \mathcal{F})$.

(a) State what it means for $\mathbb{Q}$ to be *absolutely continuous* with respect to $\mathbb{P}$, and what it means for $\mathbb{P}$ and $\mathbb{Q}$ to be *equivalent*. **[2]**

(b) Now suppose that $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$. In particular, we recall that this implies there exists an integrable random variable Z such that

$$\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}[Z\mathbb{1}_A] \qquad \text{for all} \quad A \in \mathcal{F}.$$

Suppose that there exists another random variable Y with the property that $\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}[Y\mathbb{1}_A]$ for all $A \in \mathcal{F}$.
Prove that $Y = Z$ almost surely. **[5]**

(c) Prove that

$$\mathbb{E}_{\mathbb{Q}}[X] = \mathbb{E}_{\mathbb{P}}[ZX]$$

for every integrable $\mathcal{F}$-measurable random variable X. **[4]**

(d) Now suppose that $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, so that in particular there also exists an integrable random variable $\widetilde{Z}$ such that

$$\mathbb{E}_{\mathbb{P}}[X] = \mathbb{E}_{\mathbb{Q}}\big[\widetilde{Z}X\big]$$

for every integrable $\mathcal{F}$-measurable random variable X.
Show that

$$\mathbb{E}_{\mathbb{Q}}\big[Z\widetilde{Z}\mathbb{1}_A\big] = \mathbb{E}_{\mathbb{Q}}[\mathbb{1}_A]$$

for every $A \in \mathcal{F}$, and hence deduce that $Z\widetilde{Z} = 1$ almost surely. **[4]**

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6. Let $(\xi_i)_{i\in\mathbb{N}}$ be a sequence of independent and identically distributed random variables with $\mathbb{P}(\xi_i = 1) = \mathbb{P}(\xi_i = -1) = \frac{1}{2}$, and let $\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n)$, $n \in \mathbb{Z}_+$, be the natural filtration of this sequence. Let $S = (S_n)_{n\in\mathbb{Z}_+}$ be the simple symmetric random walk given by $S_0 = 0$ and

$$S_n = \sum_{i=1}^{n} \xi_i$$

for each $n \in \mathbb{N}$. For some constant $\alpha \in \mathbb{R}$, let

$$X_n = S_n^4 - 6nS_n^2 + n(3n + \alpha)$$

for each $n \in \mathbb{Z}_+$.

For some $k \in \mathbb{N}$, let

$$T = \inf\{n \in \mathbb{Z}_+ : |S_n| = k\}.$$

Throughout this question, you may assume that T is a stopping time, and that $\mathbb{E}[T] = k^2$, so that in particular $\mathbb{P}(T < \infty) = 1$.

(a) Show that the process $X = (X_n)_{n\in\mathbb{Z}_+}$ is a martingale with respect to the filtration $(\mathcal{F}_n)_{n\in\mathbb{Z}_+}$, for some value of α which you should specify. **[6]**

(b) With the value of α found in part (a), show that

$$\mathbb{E}\big[S_{n\wedge T}^4 - 6(n \wedge T)S_{n\wedge T}^2 + (n \wedge T)\big(3(n \wedge T) + \alpha\big)\big] = 0$$

for each $n \in \mathbb{Z}_+$. **[2]**

(c) Carefully justifying your argument, deduce that

$$\mathbb{E}\big[k^4 - 6Tk^2 + T(3T + \alpha)\big] = 0.$$ **[5]**

(d) Find an expression for the value of $\mathbb{E}[T^2]$ in terms of k. **[2]**

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7. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $(X_t)_{t\in\mathbb{R}_+}$ be a real-valued submartingale with respect to a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, such that $\mathbb{E}[|X_t|] < \infty$ for every $t \in \mathbb{R}_+$.

(a) For some $t \in \mathbb{R}_+$, let t_n, $n = 1, 2, \ldots$, be a sequence of times such that $t_n \searrow t$ as $n \to \infty$.

Use the submartingale downcrossing inequality to prove that, for almost every $\omega \in \Omega$, the limit $X_{t+}(\omega) := \lim_{n\to\infty} X_{t_n}(\omega)$ exists. **[6]**

(b) Prove that $\mathbb{E}[|X_{t+}|] < \infty$ for every $t \in \mathbb{R}_+$. **[5]**

(c) Prove that $X_t \leq \mathbb{E}[X_{t+} \mid \mathcal{F}_t]$ for every $t \in \mathbb{R}_+$. **[4]**

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8. Let $(X_t^x)_{t\in\mathbb{R}_+}$ be the solution to the stochastic differential equation

$$\mathrm{d}X_t^x = -X_t^x \,\mathrm{d}t + \mathrm{d}W_t, \qquad X_0^x = x,$$

where $(W_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion on $\mathbb{R}$. For any bounded measurable function $f\colon \mathbb{R} \to \mathbb{R}$, let

$$P_t f(x) = \mathbb{E}\big[f(X_t^x)\big], \qquad t \in \mathbb{R}_+,\ x \in \mathbb{R}.$$

(a) Prove that

$$P_t f(x) = \frac{1}{\sqrt{\pi(1-e^{-2t})}} \int_{\mathbb{R}} f(y) \exp\left(- \frac{(y - e^{-t}x)^2}{1-e^{-2t}} \right) \mathrm{d}y.$$ **[4]**

(b) Prove, using the Markov property, that for any $t, s \geq 0$, and any bounded measurable function f,

$$P_t \circ P_s f = P_{t+s} f.$$ **[4]**

(c) Find a measure μ on $\mathbb{R}$ such that

$$P_t f(x) \to \int_{\mathbb{R}} f(y)\, \mu(\mathrm{d}y)$$

as $t \to \infty$ for all $x \in \mathbb{R}$. **[4]**

(d) For the measure μ obtained in part (c), prove that, for any $t \geq 0$,

$$\int_{\mathbb{R}} P_t f(x)\, \mu(\mathrm{d}x) = \int_{\mathbb{R}} f(x)\, \mu(\mathrm{d}x).$$ **[3]**
