



EXAMINATION PAPER

Examination Session: May/June	Year: 2026	Exam Code: MATH4271-WE01
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Title: Superstrings IV

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	<p>Answer all questions.</p> <p>The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.</p> <p>Write your answer in the white-covered answer booklet with barcodes.</p> <p>Begin your answer to each question on a new page.</p>
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Revision:	
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SECTION A

1. Recall the Polyakov action for a bosonic string in conformal gauge:

$$S = -\frac{T}{2} \int d^2\sigma \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu},$$

where $\eta_{\alpha\beta}$ and $\eta_{\mu\nu}$ are the Minkowski metric on the worldsheet and target space, respectively. Consider an open string with Neumann boundary conditions.

- (a) Find the equations of motion for the worldsheet fields X^μ by varying the action and keeping track of boundary conditions. [4]
 (b) Recall that the mode expansion for an open string with Neumann boundary conditions is

$$X^\mu(\tau, \sigma) = x_0^\mu + 2\alpha' p^\mu \tau + 2i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu \cos(n\sigma) e^{-in\tau}.$$

Moreover, the angular momentum is given by

$$J^{\mu\nu} = T \int_0^\pi d\sigma \left(X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu \right).$$

Show that in the classical theory

$$J^{\mu\nu} = x_0^\mu p^\nu - x_0^\nu p^\mu - i \sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu \right).$$

Hint: The following formula may be useful:

$$\int_0^\pi d\sigma \cos(n\sigma) \cos(m\sigma) = \frac{\pi}{2} (\delta_{n,m} + \delta_{n,-m}). \quad [6]$$

2. Consider the scattering of a graviton with momentum p_1^μ and two tachyons with momenta p_2^μ and p_3^μ in closed bosonic string theory.

- (a) Show that $p_1 \cdot p_2 = p_1 \cdot p_3 = 0$ and $p_2 \cdot p_3 = -4/\alpha'$. **Hint:** Use momentum conservation $\sum_{i=1}^3 p_i^\mu = 0$. [5]
 (b) The scattering amplitude is described by the following worldsheet integral:

$$\mathcal{A}_3 \propto \int \frac{\prod_{i=1}^3 d^2 z_i}{\text{Vol}(\text{SL}(2, \mathbb{C})) |z_{23}|^4} \epsilon_{\mu\nu} \left(\frac{p_2^\mu}{z_{12}} + \frac{p_3^\mu}{z_{13}} \right) \left(\frac{p_2^\nu}{\bar{z}_{12}} + \frac{p_3^\nu}{\bar{z}_{13}} \right),$$

where $z_{ij} = z_i - z_j$ and $\epsilon_{\mu\nu}$ is the polarisation of the graviton. Show that

$$\mathcal{A}_3 \propto \epsilon_{\mu\nu} (p_2^\mu - p_3^\mu) (p_2^\nu - p_3^\nu).$$

Hint: Use the $\text{SL}(2, \mathbb{C})$ symmetry and recall that $\epsilon_{\mu\nu} p_1^\mu = 0$. [5]

3. Consider the two 2×2 matrices ρ^a with $a = 1, 2$,

$$\rho^1 = \sigma_1, \quad \rho^2 = \sigma_3,$$

where the Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- (a) Show that the ρ^a matrices satisfy the two-dimensional Euclidean Clifford algebra

$$\{\rho^a, \rho^b\} = 2\delta^{ab}\mathbb{1}_2,$$

where δ^{ab} denotes the two-dimensional Kronecker delta symbol and $\mathbb{1}_2$ is the 2×2 identity matrix. [5]

- (b) Define the matrices

$$S^{ab} = \frac{1}{4}[\rho^a, \rho^b],$$

and compute the transformation

$$\exp(\theta_{ab}S^{ab})\Psi$$

of the two-dimensional Euclidean spinor $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, where $\theta_{ab} = -\theta_{ba} \in \mathbb{R}$.

Hint: Remember the Taylor series $\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. You may also wish to use $\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$ and $\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$. [5]

4. Consider a system with one set of fermionic creation/annihilation operators, \hat{b}^\dagger, \hat{b} , with canonical anti-commutation relations

$$\{\hat{b}, \hat{b}^\dagger\} = 1, \quad \{\hat{b}, \hat{b}\} = \{\hat{b}^\dagger, \hat{b}^\dagger\} = 0.$$

Define the operator

$$\hat{B} = \alpha\hat{b} + \beta\hat{b}^\dagger$$

with $\alpha, \beta \in \mathbb{R}$. Compute \hat{B}^n for all $n \in \mathbb{N}$ with $n \geq 2$. [10]

SECTION B

5. Recall that the worldsheet fields in closed bosonic string theory have the following operator product expansion:

$$X(z)X(w) = -\frac{\alpha'}{2} \ln(z-w) + \dots, \quad \tilde{X}(\bar{z})\tilde{X}(\bar{w}) = -\frac{\alpha'}{2} \ln(\bar{z}-\bar{w}) + \dots$$

where we take the target space to be one-dimensional, we write $X(z, \bar{z}) = X(z) + \tilde{X}(\bar{z})$, and \dots denotes non-singular terms. Note that in the lectures we denoted $\tilde{X}(\bar{z})$ as $X(\bar{z})$. In this case the components of the stress tensor are given by

$$T(z) = -\frac{1}{\alpha'} : \partial X(z) \partial X(z) :, \quad \tilde{T}(\bar{z}) = -\frac{1}{\alpha'} : \bar{\partial} \tilde{X}(\bar{z}) \bar{\partial} \tilde{X}(\bar{z}) :$$

- (a) State the definition of a conformal primary operator of weight (h, \tilde{h}) . [3]
 (b) Compute the singular terms in the operator product expansion of $T(z)$ with $\Phi(w) = \partial^m X(w)$, where m is any non-negative integer. [6]
 (c) Compute the singular terms in the operator product expansion of $T(z)$ with $\Phi(w) = :X(w)^m:$, where m is any positive integer. [6]

6. The Polyakov action for a closed bosonic string in a general spacetime background is

$$S = -\frac{T}{2} \int d^2\sigma \sqrt{-h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X),$$

where $h_{\alpha\beta}$ is the worldsheet metric, h is its determinant, and $G_{\mu\nu}$ is target space metric, which can be a general function of the target space coordinates X^μ .

- (a) Show that in conformal gauge the Virasoro constraints are given by

$$G_{\mu\nu} (\partial_\tau X^\mu \partial_\tau X^\nu + \partial_\sigma X^\mu \partial_\sigma X^\nu) = G_{\mu\nu} \partial_\tau X^\mu \partial_\sigma X^\nu = 0.$$

Hint: Recall that $\delta\sqrt{-h} = -\frac{1}{2}\sqrt{-h}h^{\alpha\beta}\delta h^{\alpha\beta}$ under a general variation of the worldsheet metric. [4]

- (b) Suppose the string moves in the following background:

$$G_{\mu\nu} dX^\mu dX^\nu = R^2 (-dt^2 \cosh^2 \rho + d\rho^2 + \sinh^2 \rho d\phi^2),$$

which describes three-dimensional Anti-de Sitter space with radius R . Moreover suppose that $t = \varepsilon\tau$, $\phi = \omega t$, and ρ is a general function of σ . Show that in conformal gauge

$$(\partial_\sigma \rho)^2 = \varepsilon^2 (\cosh^2 \rho - \omega^2 \sinh^2 \rho).$$

What is the maximum value of ρ that the string can have assuming $\omega^2 \geq 1$? [6]

- (c) Using the parameterisation and metric in part (b), show that the following quantities are conserved in conformal gauge:

$$E = \frac{R^2}{2\pi\alpha'} \varepsilon \int_0^{2\pi} d\sigma \cosh^2 \rho, \quad J = \frac{R^2}{2\pi\alpha'} \varepsilon \omega \int_0^{2\pi} d\sigma \sinh^2 \rho.$$

What is their physical interpretation? [5]

7. Consider the quantum mechanical Hamiltonian

$$\hat{H} = \hat{a}^\dagger \hat{a} + \hat{f}_1^\dagger \hat{f}_1 + \hat{f}_2^\dagger \hat{f}_2,$$

where a^\dagger, a are canonical bosonic creation/annihilation operators and f_i^\dagger, f_i with $i = 1, 2$ are canonical fermionic creation/annihilation operators, i.e.

$$\begin{aligned} [\hat{a}, \hat{a}^\dagger] &= 1, & [\hat{a}, \hat{a}] &= [\hat{a}^\dagger, \hat{a}^\dagger] = 0, \\ \{\hat{f}_i, \hat{f}_j^\dagger\} &= \delta_{ij}, & \{\hat{f}_i, \hat{f}_j\} &= \{\hat{f}_i^\dagger, \hat{f}_j^\dagger\} = 0, \quad i, j = 1, 2. \end{aligned}$$

Bosonic and fermionic operators commute with each other:

$$[\hat{a}, \hat{f}_i] = [\hat{a}^\dagger, \hat{f}_i] = [\hat{a}, \hat{f}_i^\dagger] = [\hat{a}^\dagger, \hat{f}_i^\dagger] = 0, \quad i = 1, 2.$$

The Fock vacuum $|0\rangle$ is the unique normalised state annihilated by \hat{a} and \hat{f}_i ,

$$\hat{a}|0\rangle = \hat{f}_i|0\rangle = 0, \quad i = 1, 2.$$

- (a) Find the eigenstates of the Hamiltonian and their corresponding eigenvalues (do not worry about obtaining normalised states). [4]
- (b) Let $(-1)^{\hat{F}}$ denote the total fermion number operator such that it anti-commutes with all fermionic operators and commutes with all bosonic ones, i.e.

$$\begin{aligned} \{(-1)^{\hat{F}}, \hat{f}_i\} &= \{(-1)^{\hat{F}}, \hat{f}_i^\dagger\} = 0, \quad i = 1, 2, \\ [(-1)^{\hat{F}}, \hat{a}] &= [(-1)^{\hat{F}}, \hat{a}^\dagger] = 0. \end{aligned}$$

If we assume that the vacuum $|0\rangle$ is bosonic, i.e. $(-1)^{\hat{F}}|0\rangle = +|0\rangle$, find all the bosonic eigenstates of the Hamiltonian, i.e. all the eigenstates $|\psi\rangle$ such that $(-1)^{\hat{F}}|\psi\rangle = +|\psi\rangle$. [3]

- (c) In the same setup as above assume now that the vacuum $|0\rangle$ is fermionic, i.e. $(-1)^{\hat{F}}|0\rangle = -|0\rangle$, find all the bosonic eigenstates of the Hamiltonian, i.e. all the eigenstates $|\psi\rangle$ such that $(-1)^{\hat{F}}|\psi\rangle = +|\psi\rangle$. [3]

- (d) Let $x \in \mathbb{R}$ be a real number with $|x| < 1$. Compute

$$\text{Tr } x^{\hat{H}} = \sum_j \frac{\langle \psi_j | x^{\hat{H}} | \psi_j \rangle}{\langle \psi_j | \psi_j \rangle},$$

where the sum runs over an orthogonal basis $\{|\psi_j\rangle\}$ of states.

Hint: Which basis should you choose as to make the calculation of $\langle \psi_j | x^{\hat{H}} | \psi_j \rangle$ the easiest possible? There is no need to compute $\langle \psi_j | \psi_j \rangle$. [5]

8. The RNS gauge-fixed Polyakov action is given by

$$S_P = -\frac{1}{4\pi\alpha'} \int_{\Sigma} d^2\sigma \eta^{ab} (\partial_a X^\mu \partial_b X^\nu + i\bar{\Psi}^\mu \rho_a \partial_b \Psi^\nu) \eta_{\mu\nu}.$$

The worldsheet coordinates are (τ, σ) and $X^\mu(\tau, \sigma)$ are D worldsheet bosonic fields, while $\Psi^\mu(\tau, \sigma)$ are D worldsheet Majorana fermions. The 2-dimensional Dirac matrices are

$$\rho^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

- (a) Compute the equations of motions for $X^\mu(\tau, \sigma)$ and $\Psi^\mu(\tau, \sigma)$. You may discard boundary terms. [4]
- (b) Given the supercurrent definition

$$J_a = \frac{1}{2} \rho^b \rho_a \Psi^\mu \partial_b X_\mu,$$

compute J_\pm in terms of the spinor components $\Psi^\mu = \begin{pmatrix} \psi_+^\mu \\ \psi_-^\mu \end{pmatrix}$. [3]

- (c) Let us fix the light-cone gauge condition

$$X^+ = \frac{X^0 + X^1}{\sqrt{2}} = p^+ \tau.$$

Use the supersymmetry transformation

$$\delta\Psi^\mu = \rho^a \partial_a X^\mu \epsilon,$$

to compute the supersymmetry variation $\delta\Psi^+$ with

$$\Psi^+ = \frac{\Psi^0 + \Psi^1}{\sqrt{2}}. \quad [4]$$

- (d) Assume that the supersymmetry parameters $\epsilon = \begin{pmatrix} \epsilon_+ \\ \epsilon_- \end{pmatrix}$ satisfy $\partial_\pm \epsilon_\pm = 0$. Combine the fermionic equations of motions you found in part (a) with the supersymmetry variation $\delta\Psi^+$ computed in part (c) to show that it is possible to choose a local supersymmetry parameter ϵ such that the fermionic light-cone gauge condition

$$\Psi^+ + \delta\Psi^+ = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

holds. [4]