



EXAMINATION PAPER

Examination Session: May/June	Year: 2026	Exam Code: MATH4281-WE01
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Title: Topics in Combinatorics IV

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	<p>Answer all questions.</p> <p>The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.</p> <p>Write your answer in the white-covered answer booklet with barcodes.</p> <p>Begin your answer to each question on a new page.</p>
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Revision:	
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SECTION A

1. Denote by $e_k(n)$ the number of Young diagrams $\lambda \vdash n$ with exactly k rows, that have **all rows of even length**, and denote by $o_k(n)$ the number of Young diagrams $\lambda \vdash n$ with exactly k rows, that have **all rows of odd length**.

- (a) Show that $e_k(n) = o_k(n - k)$ for all $n > k > 0$. [5]
- (b) Is it true that $o_k(n) = e_k(n - k)$ for all $n > k > 0$? If so, prove the identity; if not, find a correct expression for $o_k(n)$ only in terms of $e_\ell(m)$ for integers $\ell \leq k$ and $m \leq n$. [5]

2. (a) Let $w_A = 478315296 \in S_9$. Apply the Robinson–Schensted–Knuth (RSK) algorithm to compute the insertion and recording tableaux P and Q . [4]
- (b) Let R be a standard Young tableau of shape $\lambda = (3, 3, 2, 1) \vdash 9$, where

$$R = \begin{array}{|c|c|c|} \hline 1 & 4 & 5 \\ \hline 2 & 6 & 7 \\ \hline 3 & 9 & \\ \hline 8 & & \\ \hline \end{array}$$

- Find $w_B \in S_9$ that satisfies $(w_B)^2 = \text{Id}$ and whose RSK insertion tableau equals R . [4]
- (c) Find another $w_C \in S_9$, not equal to w_B , whose RSK insertion tableau also equals R . [2]

3. Let $\Gamma = \langle s_1, s_2, s_3 \mid s_i^2, (s_1 s_2)^4, (s_2 s_3)^3, (s_1 s_3)^4, s_2^7 \rangle$.
- (a) Define $\Gamma' = \langle s_1, s_3 \mid \text{all relations in } \Gamma \text{ not containing } s_2 \rangle$. Find the order of Γ' . [5]
- (b) Show that the subgroup of Γ generated by s_1 and s_3 has order 2. [5]

4. (a) Let P be the root poset of a root system Δ with disconnected Dynkin diagram shown below.



- Draw the Hasse diagram of P . [5]
- (b) Draw the Hasse diagram of the poset of order ideals of P . Identify join-irreducible elements. [5]

SECTION B

5. A Dyck path from $(0, 0)$ to $(2n, 0)$ has a *mountain* if it passes through the points $(2k - 2, 0)$, $(2k, 2)$ and $(2k + 2, 0)$ for some k satisfying $0 < k < n$. Denote by G_n the number of *mountain-free* Dyck paths of length $2n$, i.e., Dyck paths of length $2n$ that do not have a mountain. Note that $G_0 = 1$ (the trivial Dyck path does not have a mountain).

- (a) Argue carefully that $(G_n)_{n \geq 0}$ satisfies a recurrence of the form

$$C_n = G_n + \sum_{k=1}^{n-1} G_{a(k,n)} C_{b(k,n)}$$

where $a(k, n)$ and $b(k, n)$ are integer expressions in k and n that should be determined, and $(C_n)_{n \geq 0}$ is the sequence of Catalan numbers. [6]

- (b) Compute the generating function $G(x)$ of the sequence $(G_n)_{n \geq 0}$ and show that

$$G(x) = \frac{\alpha}{\beta + \gamma x^2 + \sqrt{1 - 4x}}$$

where α, β and γ are constants to be determined. [9]

Hint: You may use without proof that the generating function $C(x)$ of the Catalan numbers is

$$C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

6. Given a permutation $w \in S_n$, let $\text{cyc}(w)$ denote the number of cycles of w , and let $f_{\text{cyc}}(x) := \sum_{w \in S_n} x^{\text{cyc}(w)}$ be the generating function of cyc .

- (a) For $n = 3$, calculate $\text{cyc}(w)$ for each permutation $w \in S_3$, and show that

$$f_{\text{cyc}}(x) = (x + a_1)(x + a_2)(x + a_3),$$

where a_1, a_2 and a_3 are integers that you should determine. [3]

- (b) For integers $n \geq 1$ and $k \geq 0$, define $c(n, k) := |\{w \in S_n : \text{cyc}(w) = k\}|$ to be the number of permutations of S_n that have k cycles. Prove that

$$c(n + 1, k) = c(n, k - 1) + nc(n, k)$$

for all $n \geq 1$ and $1 \leq k \leq n + 1$. [4]

- (c) Using part (b) or otherwise, state and prove a formula for $f_{\text{cyc}}(x)$ that holds for general $n \in \mathbb{N}$. [4]

- (d) Recall that, for $w = w_1 w_2 \dots w_n \in S_n$ written in 1-line notation, w_i is a *record* of w if $w_j < w_i$ for all $j < i$. Denote by $\text{rec}(w)$ the number of records of w , and let $f_{\text{rec}}(x)$ be the generating function of rec . By considering an appropriate bijection from S_n to itself, prove that $f_{\text{rec}}(x) = f_{\text{cyc}}(x)$ for all $n \in \mathbb{N}$. [4]

7. Let (G, S) be a Coxeter system, R be the set of reflections of (G, S) , and l be the length function. Define a partial order on (G, S) as follows: for $g_1, g_2 \in G$, we say that $g_1 \leq g_2$ if there exist $t_1, \dots, t_k \in R$ such that $g_2 = g_1 t_1 \dots t_k$ and $l(g_1 t_1 \dots t_i t_{i+1}) > l(g_1 t_1 \dots t_i)$ for every $i = 1, \dots, k - 1$ (and $l(g_1 t_1) > l(g_1)$). The partial order \leq is called the *Bruhat order*.

- (a) Draw the Hasse diagram of the Bruhat order on the Weyl group of type $A_2 = I_2(3)$ (justify your answer). [5]

- (b) Let (G, S) be a Coxeter system of type A_3 . Give an example of $g_1, g_2 \in G$ with $l(g_1) \leq l(g_2)$ but g_1 being incomparable with g_2 (justify your answer). [5]

- (c) Let $G = I_2(n)$ be a dihedral group. Show that $g_1 \leq g_2$ if and only if $l(g_1) \leq l(g_2)$. [5]

8. Let Δ be the root system of type C_4 . Let Δ_l and Δ_s be the sets of long and short roots of Δ respectively.

- (a) Show that Δ_l and Δ_s are root systems and find their types. [5]

- (b) Compute the Coxeter number of Δ_s . [5]

- (c) Find the exponents of the Weyl group of Δ_s . [5]