Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH42920-WE01

Title: Functional Analysis and Applications V

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

SECTION A

1. (a) Show that

$$\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix} \right\}$$

is a Hamel basis of $\mathbb{R}^2$. **[5]**

(b) Find a Schauder basis of $\ell^2(\mathbb{N})$ that is not orthonormal. Justify your choice. **[5]**

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2. Let $X = C[0,1]$ be equipped with the norm $\|\cdot\|_\infty$ and consider the subspace $Y = \text{span}\{y\}$, where $y(t) = 1$ for all $t \in [0,1]$. Define $f : Y \to \mathbb{C}$ by

$$f(\alpha y) = \alpha, \quad \alpha \in \mathbb{C}.$$

(a) Show that $f \in Y^*$ and calculate the operator norm $\|f\|_{Y^*}$. **[4]**

(b) Prove that there exists $F \in X^*$ with $F(x) = f(x)$ for all $x \in Y$ and $\|F\|_{X^*} = \|f\|_{Y^*}$. Is F unique? Justify your answer. **[6]**

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3. We consider the two absolutely continuous probability measures on the real line given via their densities f, g,

$$f(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & \text{otherwise}, \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1, & x \in [1,2] \\ 0, & \text{otherwise}. \end{cases}$$

(a) Using the definition of the pushforward operator, show that the function $T : [0,1] \to [1,2]$ defined as $T(x) = -x + 2$ is a transport map between the two densities f and g, that is $T_\sharp f = g$. **[3]**

(b) Show that T defined in Question **3.**a cannot be an optimal transport map for the quadratic cost function $c : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, defined as $c(x,y) = |x-y|^2$. **[3]**

(c) Find the optimal transport map between f and g for the quadratic cost, and give a proof of why it is optimal. **[4]**

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4. Let $d \in \mathbb{N}$ and let $\Omega \subset \mathbb{R}^d$ be a compact set and let $\mu \in \mathcal{P}(\Omega)$ be a probability measure. Let $\nu = \delta_{\vec{x}_0}$ with $\vec{x}_0 \in \Omega$ be the Dirac delta mass concentrated at $\vec{x}_0$. Let $c : \Omega \times \Omega \to \mathbb{R}$ be a continuous function and let

$$\Pi(\mu, \nu) := \{\gamma \in \mathcal{P}(\Omega \times \Omega) : (p^x)_\sharp \gamma = \mu, (p^y)_\sharp \gamma = \nu\}$$

denote the usual set of transport plans between μ and ν.

(a) Show that the optimal transport problem from μ to ν

$$\mathcal{T}_c(\mu, \nu) := \inf \left\{ \iint_{\Omega \times \Omega} c(\vec{x}, \vec{y}) d\gamma(\vec{x}, \vec{y}) : \gamma \in \Pi(\mu, \nu) \right\},$$

has a unique solution γ_{opt}, which is concentrated on the graph of a map $T : \Omega \to \Omega$, i.e. $\gamma_{opt} = (\mathrm{id}, T)_\sharp \mu$. Find this map. Here $\mathrm{id} : \mathbb{R}^d \to \mathbb{R}^d$ stands for the identity map, i.e. $\mathrm{id}(\vec{x}) = \vec{x}$. **[4]**

(b) Consider the optimal transport problem from ν to μ, i.e.

$$\mathcal{T}_c(\nu, \mu) := \inf \left\{ \iint_{\Omega \times \Omega} c(\vec{x}, \vec{y}) d\gamma(\vec{x}, \vec{y}) : \gamma \in \Pi(\nu, \mu) \right\}.$$

Show that this also has a unique solution, but this cannot be induced by a map, unless $\mathrm{spt}(\mu)$ is a singleton. **[3]**

(c) Now let $\mu := \frac{1}{4}\delta_{\vec{y}_1} + \frac{3}{4}\delta_{\vec{y}_2}$, where $\vec{y}_1, \vec{y}_2 \in \Omega$ are distinct points. Compute the optimal values $\mathcal{T}_c(\mu, \nu)$ and $\mathcal{T}_c(\nu, \mu)$. **[3]**

SECTION B

5. Consider the space of convergent sequences

$$c = \{x = (x_n)_{n\in\mathbb{N}} \subset \mathbb{C} : \lim_{n\to\infty} x_n \text{ exists}\}.$$

(a) Prove that c is not dense in $\ell^\infty(\mathbb{N})$. [*Hint:* Show that $y = (y_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$ with $y_n = (-1)^n$ is not approximated by elements in c.] **[6]**

(b) Using that $\ell^\infty(\mathbb{N})$ can be identified with the dual space of $\ell^1(\mathbb{N})$ (which you can use without proof), show that c is weak-$*$ dense in $\ell^\infty(\mathbb{N})$, i.e. for every $x \in \ell^\infty(\mathbb{N})$ there exists $(x^{(N)})_{N\in\mathbb{N}} \subset c$ with $x^{(N)} \overset{w-*}{\to} x$. **[9]**

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6. (a) State the Closed Graph Theorem. **[3]**

(b) Let $T : \mathcal{D}(T) \subset \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the linear operator defined by

$$(Tx)_n = nx_n, \quad \mathcal{D}(T) = \left\{ x = (x_n)_{n\in\mathbb{N}} \subset \mathbb{C} : \sum_{n\in\mathbb{N}} |nx_n|^2 < \infty \right\}.$$

Show that T is bijective and its inverse $T^{-1} : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ is bounded. **[5]**

(c) Show that the operator T in Question **6.**b is closed but not bounded. Why is this not a contradiction to the Closed Graph Theorem? **[7]**

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7. For $d \in \mathbb{N}$ and $r > 0$, let $\alpha_d(r)$ stand for the d-dimensional volume of the d-dimensional ball with radius r. We denote the ball centred at the origin with radius r by

$$B_r := \left\{\vec{x} \in \mathbb{R}^d : \ |\vec{x}| \le r\right\}.$$

Let $\mu_0 = \frac{1}{\alpha_d(1)}\mathcal{L}^d|_{B_1}$ be the normalised d-dimensional Lebesgue measure restricted to B_1. Furthermore, define $T : \mathbb{R}^d \to \mathbb{R}^d$ as $T(\vec{x}) = 2\vec{x}$ and for $t \in [0,1]$ define

$$\mu_t := ((1-t)\mathrm{id} + tT)_\sharp \, \mu_0,$$

where $\mathrm{id} : \mathbb{R}^d \to \mathbb{R}^d$ stands for the identity map, i.e. $\mathrm{id}(\vec{x}) = \vec{x}$.

(a) Show that the map $T_t := (1-t)\mathrm{id} + tT$ is invertible for all $t \in [0,1]$ and compute $(T_t)^{-1}$. **[2]**

(b) Determine the density of the measure μ_t, for all $t \in (0,1]$. [*Hint:* use the Monge–Ampère equation.] **[4]**

(c) Find the optimal transport map in the definition of $W_2(\mu_0, \mu_1)$, which is the 2-Wasserstein distance between μ_0 and μ_1. **[3]**

(d) Define the vector field $\vec{v} : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$ as

$$\vec{v}(t,\cdot) = (T - \mathrm{id}) \circ (T_t)^{-1}.$$

Show that the pair $(\mu_t, \vec{v}(t,\cdot))_{t\in[0,1]}$ is a distributional solution for the continuity equation

$$\partial_t \mu_t + \mathrm{div}(\vec{v}(t,\cdot)\mu_t) = 0$$

on $(0,1) \times \mathbb{R}^d$. **[3]**

(e) Find a constant speed W_2-geodesic between μ_0 and μ_1, where W_2 stands for the 2-Wasserstein distance. **[3]**

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8. Let $d \in \mathbb{N}$, $\varepsilon \geq 0$ and let $V : \mathbb{R}^d \to \mathbb{R}$ be a convex potential function of at most quadratic growth at infinity. Consider the minimisation problem

$$\min \left\{ \int_{\mathbb{R}^d} V(\vec{x}) d\rho(\vec{x}) + \varepsilon \int_{\mathbb{R}^d} \rho(\vec{x}) \log \rho(\vec{x}) d\vec{x} : \quad \rho \in \mathcal{P}_2(\mathbb{R}^d) \right\}.$$

Here, $\mathcal{P}_2(\mathbb{R}^d)$ stands for the set of Borel probability measures supported on $\mathbb{R}^d$ with finite second moment. The logarithmic entropy functional is defined to take the value $+\infty$ for any measure, which is not absolutely continuous with respect to the d-dimensional Lebesgue measure. Throughout this question we assume that this variational problem has a solution, for all $\varepsilon \geq 0$. We will denote this minimiser by ρ_ε.

(a) Show that the minimiser ρ_ε is unique, for all $\varepsilon > 0$. **[2]**

(b) Write down the first order optimality condition satisfied by the minimiser, and give a formula for the minimiser. **[5]**

(c) Determine the unique minimiser ρ_0 when $V(\vec{x}) = |\vec{x}|^2$ and $\varepsilon = 0$. **[2]**

(d) In the case of $V(\vec{x}) := |\vec{x}|^2$, show that ρ_ε converges in the sense of distributions to ρ_0 (the minimiser obtained in Question **8.**c) when $\varepsilon \to 0$. [*Hint:* you may use without proof that $\int_{\mathbb{R}^d} e^{-|\vec{x}|^2} d\vec{x} = \pi^{d/2}$.] **[6]**
