Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH43320-WE01

Title:
Ergodic Theory V

Time:	3 hours	
Additional Material provided:	Formula sheet	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

SECTION A

1. In what follows, let $S^1 = \{z \in \mathbb{C} \colon |z| = 1\}$ denote the unit circle and let I denote the interval $[-1, 1]$. Consider the topological dynamical systems (S^1, f) and (I, g), where $f(z) = z^2$ $(z \in S^1)$, and $g(x) = 2x^2 - 1$ $(x \in I)$.

(a) Show that

$$\pi \colon S^1 \to I, \qquad z \mapsto \mathrm{Re}(z),$$

is a topological factor map from (S^1, f) to (I, g). Here $\mathrm{Re}(z)$ denotes the real part of $z \in S^1$. **[6]**

(b) Can (S^1, f) and (I, g) be topologically isomorphic? Justify your answer. **[4]**

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2. In the following, $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$, and $(\mathbb{T}^1, \alpha)$ is an irrational rotation.

(a) Show that $(\mathbb{T}^1, \alpha)$ is ergodic with respect to Lebesgue measure λ.
Hint: You may use without proof that for each $f \in L_2(\mathbb{T}^1)$, there are unique coefficients a_k $(k \in \mathbb{Z})$ such that

$$f(x) = \sum_{k \in \mathbb{Z}} a_k \, e^{2\pi i k x} \text{ for } \lambda\text{–a.e. } x \in \mathbb{T}^1.$$

[8]

(b) Prove that ergodicity of $(\mathbb{T}^1, \alpha)$ with respect to λ implies minimality of $(\mathbb{T}^1, \alpha)$. You may use any results proved in the lectures. **[2]**

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3. Let A and P be the following matrices,

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Recall that we denote by (Σ_A^+, σ) the subshift of finite type corresponding to A.

(a) Let μ_P be the Parry measure on Σ_A^+ associated to P. Compute the μ_P-measure of every cylinder set of length two in Σ_A^+.
Hint: You may use that $(4,4,6)P = (4,4,6)$. **[4]**

(b) Justify why μ_P is strong mixing.
Hint: You may use a result from lectures stating that strong mixing of Markov measures is implied by a property of A. **[2]**

(c) Let $f : \Sigma_A^+ \to \mathbb{R}$ be continuous and set

$$f_N = \frac{1}{2N} \sum_{n=0}^{2N-1} (-1)^n f \circ \sigma^n.$$

Argue that for μ_P almost all $x \in \Sigma_A^+$ we have $\lim_{N\to\infty} f_N(x) = 0$.
Hint: You may use Birkhoff's ergodic theorem here. **[4]**

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4. Let S be a rotation of $\mathbb{T}^1$ and let E_2 be the doubling map on $\mathbb{T}^1$.

(a) Show that the topological entropy of S is zero.
Hint: state and use one of the equivalent characterizations of topological entropy using covering sets, separating sets, or spanning sets. **[6]**

(b) Evaluate the topological entropy of the map $T = S^{-1} \circ E_2 \circ S$ and briefly justify your answer.
Hint: Here you may use the known value of the topological entropy of E_2 *as given in lectures.* **[4]**

SECTION B

5. In what follows, (X, T) is a topological dynamical system.

(a) Suppose that for each continuous function $f\colon X \to \mathbb{R}$, there is a constant $c \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = c$$

for all $x \in X$. Show that (X, T) is uniquely ergodic.

Hint: You may use without proof that two Borel probability measures μ_1 and μ_2 on X coincide if and only if $\int_X f\, d\mu_1 = \int_X f\, d\mu_2$ for each continuous function $f\colon X \to \mathbb{R}$. **[6]**

(b) Suppose now that (X, T) is uniquely ergodic. Prove that (X, T) has exactly one minimal subset. **[5]**

(c) Is it possible that for each continuous function $f\colon X \to \mathbb{R}$, the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x)$$

exists for each $x \in X$ even if (X, T) is not uniquely ergodic? Either give an appropriate example or prove why such (X, T) cannot exist. **[4]**

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6. Consider the *steep tent map* $T\colon [0,1] \to \mathbb{R}$ defined by

$$T(x) = \begin{cases} 3x & \text{if } 0 \leq x \leq 1/2, \\ 3(1-x) & \text{if } 1/2 < x \leq 1. \end{cases}$$

Notice that points around the centre of the interval are mapped outside of $[0,1]$ under the map T; we say that such points *escape*. Let Λ be the set of points that *never* escape, that is, $x \in \Lambda$ if and only if $T^n(x) \in [0,1]$ for all $n \geq 0$.

Note: In what follows, you can use without proof that every $x \in [0,1]$ has a ternary expansion *of the form $x = \sum_{n=1}^{\infty} x_n 3^{-n}$, where $(x_n)_{n\in\mathbb{N}}$ is a sequence with $x_n \in \{0,1,2\}$ for all n. You may further use the fact that for any $x \in [0,1]$, the ℓ-th entry x_ℓ in its ternary expansion is uniquely determined unless $x_n = 0$ for all $n > \ell$ or $x_n = 2$ for all $n > \ell$.*

(a) Show that given $x \in [0,1]$, $T(x) \in [0,1]$ if and only if x admits a ternary expansion with $x_1 \in \{0,2\}$. **[5]**

(b) Given $x \in [0,1]$, one can show that $x \in \Lambda$ if and only if there is a ternary expansion $x = \sum_{n=1}^{\infty} x_n 3^{-n}$ with $x_n \in \{0,2\}$ for all $n \in \mathbb{N}$. Prove the "if" direction of that statement. **[5]**

(c) Let $\hat{\sigma} : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ be the *twisted shift map* defined by

$$\hat{\sigma}(y_1, y_2, y_3, \dots) = \begin{cases} (y_2, y_3, y_4, \dots) & \text{if } y_1 = 0, \\ (1-y_2, 1-y_3, 1-y_4, \dots) & \text{if } y_1 = 1. \end{cases}$$

Show that $(\{0,1\}^{\mathbb{N}}, \hat{\sigma})$ and $(\Lambda, T|_\Lambda)$ are topologically isomorphic.
Note: We consider $\{0,1\}^{\mathbb{N}}$ equipped with the Cantor metric. Further, you may use the full if and only if statement for Λ from part (b). **[5]**

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7. Let $X = [0,1]$ and let $T : X \to X$ be the Gauss map,

$$T(x) = \begin{cases} \frac{1}{x} - \lfloor \frac{1}{x} \rfloor, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Let μ be the Gauss measure on X, given by

$$\mu(A) = \frac{1}{\log 2} \int_A \frac{1}{1+x}\, dx,$$

for each Borel measurable set $A \subset X$. Let $\mathrm{Leb}_{\mathbb{R}}$ denote the Lebesgue measure on the real line $\mathbb{R}$.

(a) Show that for every Borel measurable set $A \subset X$ we have

$$\frac{1}{2\log 2}\mathrm{Leb}_{\mathbb{R}}(A) \leq \mu(A) \leq \frac{1}{\log 2}\mathrm{Leb}_{\mathbb{R}}(A).$$

[3]

(b) For $n = 1, 2, \ldots$ let $A_n = (1/(n+1), 1/n]$, and set $A_0 = \{0\}$. Let α be the measurable partition consisting of all A_n, namely,

$$\alpha = \{A_0, A_1, A_2 \ldots\}.$$

Show that $H_\mu(\alpha) < \infty$. **[5]**

(c) Deduce that $h_\mu(T) < \infty$.
Hint: You may use that α is a strong generator for T. **[7]**

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8. Let $\widetilde{A} : \mathbb{T}^2 \to \mathbb{T}^2$ be the toral automorphism defined by the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Note: Recall that points in $\mathbb{T}^2$ *are equivalence classes*

$$\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] = \begin{pmatrix} x \\ y \end{pmatrix} + \mathbb{Z}^2,$$

for $x, y \in \mathbb{R}$*; and recall that each equivalence class has a representative with* $x, y \in [0, 1)$*. Recall that Lebesgue measure* m_{Leb} *on* $\mathbb{T}^2$ *is invariant for* $\widetilde{A}$*.*

(a) Evaluate the image of $\left[\begin{pmatrix} x \\ y \end{pmatrix}\right]$ under $(\widetilde{A})^n$ for $n \in \mathbb{N}$ and $x, y \in [0, 1)$. **[2]**

(b) Show that for each $x \in [0, 1)$ the set $L_x \subset \mathbb{T}^2$ given by

$$L_x = \left\{ \left[\begin{pmatrix} x \\ t \end{pmatrix}\right] : t \in \mathbb{R} \right\}$$

is an invariant set for $\widetilde{A}$. **[2]**

(c) Show that m_{Leb} is not ergodic for $\widetilde{A}$. **[4]**

(d) Show that for any ergodic invariant Borel probability measure μ there is some L_x with $\mu(L_x) = 1$. **[5]**

Hint: You may use "continuity of measure" which states that for any sequence of measurable sets $B_1, B_2, \ldots$ *with* $B_{n+1} \subset B_n$ *for each* n *we have that* $\lim_{n\to\infty} \mu(B_n) = \mu(\bigcap_{n=1}^{\infty} B_n)$*.*

(e) Deduce that $h_\mu(\widetilde{A}) = 0$ for any ergodic invariant Borel probability measure μ. **[2]**
