Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH4341-WE01

Title: Spatio-Temporal Statistics

Time:	2 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

SECTION A

1. (a) Consider an m-dimensional multivariate stationary time series process, $\mathbf{X}_t$, $t \in \mathbb{Z}$. Since the process is stationary, $\mathbb{C}\text{ov}(\mathbf{X}_s, \mathbf{X}_{s+t})$ will be independent of $s \in \mathbb{Z}$ for any $t \in \mathbb{Z}$, and we can define

$$\Gamma_t = \mathbb{C}\text{ov}(\mathbf{X}_s, \mathbf{X}_{s+t}), \quad \forall t \in \mathbb{Z}.$$

(i) Show that $\Gamma_{-t} = \Gamma_t^\top, \ \forall t \in \mathbb{Z}$. **[2]**

(ii) Briefly discuss the implications of this for the time-reversibility of univariate and multivariate stationary *Gaussian* time series models. **[2]**

(b) Consider the first-order m-dimensional vector auto-regressive, VAR(1), model

$$\mathbf{X}_t = \Phi\mathbf{X}_{t-1} + \boldsymbol{\varepsilon}_t, \quad \boldsymbol{\varepsilon}_t \sim \mathcal{N}(\mathbf{0}, \Sigma), \quad \forall t \in \mathbb{Z},$$

for some $m \times m$ matrix Φ and strictly positive definite matrix Σ.

(i) What condition is required on this model in order to ensure stationarity of this process? **[1]**

(ii) Assuming stationarity, write down an equation satisfied by the stationary variance matrix, V. **[2]**

(iii) Deduce the form of the auto-covariance function, Γ_t, $t \geq 0$, as an explicit function of V, Φ and t. **[3]**

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2. A Gaussian Markov random field (GMRF), $\mathbf{X}$, of dimension n, has distribution

$$\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \mathsf{Q}^{-1}),$$

where $\boldsymbol{\mu}$ is an n-vector and Q is a $n \times n$ sparse precision matrix. The latent field is not directly observed, but indirectly, via m-vector $\mathbf{Y}$, modelled as

$$(\mathbf{Y}|\mathbf{X} = \mathbf{x}) \sim \mathcal{N}(\mathsf{F}\mathbf{x}, \Lambda^{-1}),$$

where $m \times n$ observation matrix F and $m \times m$ precision matrix Λ are both sparse.

(a) By multiplying appropriate normal densities and completing the square (or otherwise), show that

$$(\mathbf{X}|\mathbf{Y} = \mathbf{y}) \sim \mathcal{N}(\tilde{\boldsymbol{\mu}}, \tilde{\mathsf{Q}}^{-1}),$$

where $\tilde{\mathsf{Q}} = \mathsf{Q} + \mathsf{F}^\top\Lambda\mathsf{F}$ and you should deduce the corresponding expression for $\tilde{\boldsymbol{\mu}}$. **[5]**

(b) After noting that $\tilde{\mathsf{Q}}$ is sparse, explain how to compute $\tilde{\boldsymbol{\mu}}$ using only sparse matrix operations (without creating any dense matrices). **[3]**

(c) Assuming a source of iid standard normal random quantities, explain carefully how to generate iid samples from $(\mathbf{X}|\mathbf{Y} = \mathbf{y})$ using only sparse matrix operations. **[2]**

SECTION B

3. Consider the second-order auto-regressive model

$$X_t = \frac{\sqrt{3}}{2} X_{t-1} - \frac{3}{4} X_{t-2} + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0,1), \quad \forall t \in \mathbb{Z}.$$

(a) (i) Confirm that this model is stationary. **[2]**

(ii) Compute the auto-correlation function, ρ_t, of this process. Your final expression should be an explicit function of t and should not involve the imaginary unit, i. **[4]**

(b) (i) Obtain the stationary variance of the process. **[2]**

(ii) Without further computation, write down the partial auto-correlation function for this process. **[2]**

(c) (i) Compute the spectral density function, $S(\nu)$, for this process as a function of the (raw) frequency $\nu \in [0, \frac{1}{2}]$. Your final expression should not involve the imaginary unit, i. **[3]**

(ii) Without further computation, at roughly what value of ν would you expect a peak in the spectral density to occur? Justify your answer. **[2]**

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4. A second-order random field, $Z(\mathbf{s})$, $\mathbf{s} \in \mathcal{D} = \mathbb{R}^2$ is characterised by the semi-variogram function,

$$\gamma(\mathbf{h}) = \frac{1}{2}\mathbb{V}\text{ar}[Z(\mathbf{h}) - Z(\mathbf{0})], \quad \forall \mathbf{h} \in \mathcal{D}.$$

(a) Explain what is meant by an *intrinsically stationary* random field. **[2]**

(b) Show that if the value of an intrinsically stationary random field is known at the origin, say $Z(\mathbf{0}) = \mu$, then the semi-variogram completely determines the covariance function,

$$C(\mathbf{s}, \mathbf{s}') = \mathbb{C}\text{ov}[Z(\mathbf{s}), Z(\mathbf{s}')], \quad \forall \mathbf{s}, \mathbf{s}' \in \mathcal{D}.$$ **[3]**

(c) Deduce the semi-variogram function, $\tilde{\gamma}(\cdot)$ of the random field

$$Y(\mathbf{s}) = aZ(b\mathbf{s}), \quad \forall \mathbf{s} \in \mathcal{D}$$

for given $a, b > 0$, in terms of the semi-variogram function $\gamma(\cdot)$, of $Z(\cdot)$. **[3]**

(d) An intrinsically stationary random field, $Z(\mathbf{s})$ is said to be *self-affine* if for every $b > 0$ there exists an $a > 0$ such that $Z(\mathbf{s})$ and $aZ(b\mathbf{s})$ have the same (semi-)variogram. *Fractional Brownian motion* has semi-variogram

$$\gamma(\mathbf{h}) = \frac{1}{2}\|\mathbf{h}\|^{2H}, \quad \forall \mathbf{h} \in \mathcal{D},$$

for given $H \in (0, 1)$. Show that fractional Brownian motion is self-affine. **[2]**

(e) Assuming that $Z(\mathbf{0}) = 0$, show that the covariance function of fractional Brownian motion takes the form

$$C(\mathbf{s}, \mathbf{s}') = \frac{1}{2}\left(\|\mathbf{s}\|^{2H} + \|\mathbf{s}'\|^{2H} - \|\mathbf{s} - \mathbf{s}'\|^{2H}\right).$$ **[2]**

(f) Compute the covariance between increments,

$$\mathbb{C}\text{ov}[Z(\mathbf{s}) - Z(\mathbf{s} - \mathbf{h}), Z(\mathbf{s} + \mathbf{h}) - Z(\mathbf{s})],$$

and show that it is independent of $\mathbf{s}$. **[3]**
