



EXAMINATION PAPER

Examination Session: May/June	Year: 2026	Exam Code: MATH4361-WE01
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Title: Ergodic Theory IV

Time:	3 hours	
Additional Material provided:	Formula sheet	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	<p>Answer all questions.</p> <p>The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.</p> <p>Write your answer in the white-covered answer booklet with barcodes.</p> <p>Begin your answer to each question on a new page.</p>
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Revision:	
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SECTION A

1. In what follows, let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ denote the unit circle and let I denote the interval $[-1, 1]$. Consider the topological dynamical systems (S^1, f) and (I, g) , where $f(z) = z^2$ ($z \in S^1$), and $g(x) = 2x^2 - 1$ ($x \in I$).

(a) Show that

$$\pi: S^1 \rightarrow I, \quad z \mapsto \operatorname{Re}(z),$$

is a topological factor map from (S^1, f) to (I, g) . Here $\operatorname{Re}(z)$ denotes the real part of $z \in S^1$.

[6]

(b) Can (S^1, f) and (I, g) be topologically isomorphic? Justify your answer.

[4]

2. In the following, $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$, and (\mathbb{T}^1, α) is an irrational rotation.

(a) Show that (\mathbb{T}^1, α) is ergodic with respect to Lebesgue measure λ .

Hint: You may use without proof that for each $f \in L_2(\mathbb{T}^1)$, there are unique coefficients a_k ($k \in \mathbb{Z}$) such that

$$f(x) = \sum_{k \in \mathbb{Z}} a_k e^{2\pi i k x} \text{ for } \lambda\text{-a.e. } x \in \mathbb{T}^1.$$

[8]

(b) Prove that ergodicity of (\mathbb{T}^1, α) with respect to λ implies minimality of (\mathbb{T}^1, α) . You may use any results proved in the lectures.

[2]

3. Let A and P be the following matrices,

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Recall that we denote by (Σ_A^+, σ) the subshift of finite type corresponding to A .

(a) Let μ_P be the Parry measure on Σ_A^+ associated to P . Compute the μ_P -measure of every cylinder set of length two in Σ_A^+ .

Hint: You may use that $(4, 4, 6)P = (4, 4, 6)$.

[4]

(b) Justify why μ_P is strong mixing.

Hint: You may use a result from lectures stating that strong mixing of Markov measures is implied by a property of A .

[2]

(c) Let $f : \Sigma_A^+ \rightarrow \mathbb{R}$ be continuous and set

$$f_N = \frac{1}{2^N} \sum_{n=0}^{2^N-1} (-1)^n f \circ \sigma^n.$$

Argue that for μ_P almost all $x \in \Sigma_A^+$ we have $\lim_{N \rightarrow \infty} f_N(x) = 0$.

Hint: You may use Birkhoff's ergodic theorem here.

[4]

4. Let S be a rotation of \mathbb{T}^1 and let E_2 be the doubling map on \mathbb{T}^1 .

(a) Show that the topological entropy of S is zero.

Hint: state and use one of the equivalent characterizations of topological entropy using covering sets, separating sets, or spanning sets.

[6]

(b) Evaluate the topological entropy of the map $T = S^{-1} \circ E_2 \circ S$ and briefly justify your answer.

Hint: Here you may use the known value of the topological entropy of E_2 as given in lectures.

[4]

SECTION B

5. In what follows, (X, T) is a topological dynamical system.

- (a) Suppose that for each continuous function $f: X \rightarrow \mathbb{R}$, there is a constant $c \in \mathbb{R}$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = c$$

for all $x \in X$. Show that (X, T) is uniquely ergodic.

Hint: You may use without proof that two Borel probability measures μ_1 and μ_2 on X coincide if and only if $\int_X f d\mu_1 = \int_X f d\mu_2$ for each continuous function $f: X \rightarrow \mathbb{R}$.

[6]

- (b) Suppose now that (X, T) is uniquely ergodic. Prove that (X, T) has exactly one minimal subset.

[5]

- (c) Is it possible that for each continuous function $f: X \rightarrow \mathbb{R}$, the limit

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x)$$

exists for each $x \in X$ even if (X, T) is not uniquely ergodic? Either give an appropriate example or prove why such (X, T) cannot exist.

[4]

6. Consider the *steep tent map* $T: [0, 1] \rightarrow \mathbb{R}$ defined by

$$T(x) = \begin{cases} 3x & \text{if } 0 \leq x \leq 1/2, \\ 3(1-x) & \text{if } 1/2 < x \leq 1. \end{cases}$$

Notice that points around the centre of the interval are mapped outside of $[0, 1]$ under the map T ; we say that such points *escape*. Let Λ be the set of points that *never* escape, that is, $x \in \Lambda$ if and only if $T^n(x) \in [0, 1]$ for all $n \geq 0$.

Note: In what follows, you can use without proof that every $x \in [0, 1]$ has a ternary expansion of the form $x = \sum_{n=1}^{\infty} x_n 3^{-n}$, where $(x_n)_{n \in \mathbb{N}}$ is a sequence with $x_n \in \{0, 1, 2\}$ for all n . You may further use the fact that for any $x \in [0, 1]$, the ℓ -th entry x_ℓ in its ternary expansion is uniquely determined unless $x_n = 0$ for all $n > \ell$ or $x_n = 2$ for all $n > \ell$.

(a) Show that given $x \in [0, 1]$, $T(x) \in [0, 1]$ if and only if x admits a ternary expansion with $x_1 \in \{0, 2\}$. [5]

(b) Given $x \in [0, 1]$, one can show that $x \in \Lambda$ if and only if there is a ternary expansion $x = \sum_{n=1}^{\infty} x_n 3^{-n}$ with $x_n \in \{0, 2\}$ for all $n \in \mathbb{N}$. Prove the "if" direction of that statement. [5]

(c) Let $\hat{\sigma}: \{0, 1\}^{\mathbb{N}} \rightarrow \{0, 1\}^{\mathbb{N}}$ be the *twisted shift map* defined by

$$\hat{\sigma}(y_1, y_2, y_3, \dots) = \begin{cases} (y_2, y_3, y_4, \dots) & \text{if } y_1 = 0, \\ (1 - y_2, 1 - y_3, 1 - y_4, \dots) & \text{if } y_1 = 1. \end{cases}$$

Show that $(\{0, 1\}^{\mathbb{N}}, \hat{\sigma})$ and $(\Lambda, T|_{\Lambda})$ are topologically isomorphic.

Note: We consider $\{0, 1\}^{\mathbb{N}}$ equipped with the Cantor metric. Further, you may use the full if and only if statement for Λ from part (b). [5]

7. Let $X = [0, 1]$ and let $T : X \rightarrow X$ be the Gauss map,

$$T(x) = \begin{cases} \frac{1}{x} - \lfloor \frac{1}{x} \rfloor, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Let μ be the Gauss measure on X , given by

$$\mu(A) = \frac{1}{\log 2} \int_A \frac{1}{1+x} dx,$$

for each Borel measurable set $A \subset X$. Let $\text{Leb}_{\mathbb{R}}$ denote the Lebesgue measure on the real line \mathbb{R} .

(a) Show that for every Borel measurable set $A \subset X$ we have

$$\frac{1}{2 \log 2} \text{Leb}_{\mathbb{R}}(A) \leq \mu(A) \leq \frac{1}{\log 2} \text{Leb}_{\mathbb{R}}(A).$$

[3]

(b) For $n = 1, 2, \dots$ let $A_n = (1/(n+1), 1/n]$, and set $A_0 = \{0\}$. Let α be the measurable partition consisting of all A_n , namely,

$$\alpha = \{A_0, A_1, A_2, \dots\}.$$

Show that $H_{\mu}(\alpha) < \infty$.

[5]

(c) Deduce that $h_{\mu}(T) < \infty$.

Hint: You may use that α is a strong generator for T .

[7]

8. Let $\tilde{A} : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be the toral automorphism defined by the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Note: Recall that points in \mathbb{T}^2 are equivalence classes

$$\left[\begin{pmatrix} x \\ y \end{pmatrix} \right] = \begin{pmatrix} x \\ y \end{pmatrix} + \mathbb{Z}^2,$$

for $x, y \in \mathbb{R}$; and recall that each equivalence class has a representative with $x, y \in [0, 1)$. Recall that Lebesgue measure m_{Leb} on \mathbb{T}^2 is invariant for \tilde{A} .

(a) Evaluate the image of $\left[\begin{pmatrix} x \\ y \end{pmatrix} \right]$ under $(\tilde{A})^n$ for $n \in \mathbb{N}$ and $x, y \in [0, 1)$. [2]

(b) Show that for each $x \in [0, 1)$ the set $L_x \subset \mathbb{T}^2$ given by

$$L_x = \left\{ \left[\begin{pmatrix} x \\ t \end{pmatrix} \right] : t \in \mathbb{R} \right\}$$

is an invariant set for \tilde{A} . [2]

(c) Show that m_{Leb} is not ergodic for \tilde{A} . [4]

(d) Show that for any ergodic invariant Borel probability measure μ there is some L_x with $\mu(L_x) = 1$. [5]

Hint: You may use “continuity of measure” which states that for any sequence of measurable sets B_1, B_2, \dots with $B_{n+1} \subset B_n$ for each n we have that $\lim_{n \rightarrow \infty} \mu(B_n) = \mu(\bigcap_{n=1}^{\infty} B_n)$.

(e) Deduce that $h_{\mu}(\tilde{A}) = 0$ for any ergodic invariant Borel probability measure μ . [2]