



EXAMINATION PAPER

Examination Session: May/June	Year: 2026	Exam Code: MATH4371-WE01
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Title: Functional Analysis and Applications IV
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Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	<p>Answer all questions.</p> <p>The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply.</p> <p>Write your answer in the white-covered answer booklet with barcodes.</p> <p>Begin your answer to each question on a new page.</p>
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Revision:	
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SECTION A

1. (a) Show that

$$\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix} \right\}$$

is a Hamel basis of \mathbb{R}^2 .

[5]

- (b) Find a Schauder basis of
- $\ell^2(\mathbb{N})$
- that is not orthonormal. Justify your choice.

[5]

2. Let
- $X = C[0, 1]$
- be equipped with the norm
- $\|\cdot\|_\infty$
- and consider the subspace
- $Y = \text{span}\{y\}$
- , where
- $y(t) = 1$
- for all
- $t \in [0, 1]$
- . Define
- $f : Y \rightarrow \mathbb{C}$
- by

$$f(\alpha y) = \alpha, \quad \alpha \in \mathbb{C}.$$

- (a) Show that
- $f \in Y^*$
- and calculate the operator norm
- $\|f\|_{Y^*}$
- .

[4]

- (b) Prove that there exists
- $F \in X^*$
- with
- $F(x) = f(x)$
- for all
- $x \in Y$
- and
- $\|F\|_{X^*} = \|f\|_{Y^*}$
- . Is
- F
- unique? Justify your answer.

[6]

3. We consider the two absolutely continuous probability measures on the real line given via their densities
- f, g
- ,

$$f(x) = \begin{cases} 1, & x \in [0, 1] \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1, & x \in [1, 2] \\ 0, & \text{otherwise.} \end{cases}$$

- (a) Using the definition of the pushforward operator, show that the function
- $T : [0, 1] \rightarrow [1, 2]$
- defined as
- $T(x) = -x + 2$
- is a transport map between the two densities
- f
- and
- g
- , that is
- $T_{\#}f = g$
- .

[3]

- (b) Show that
- T
- defined in Question 3.a cannot be an optimal transport map for the quadratic cost function
- $c : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$
- , defined as
- $c(x, y) = |x - y|^2$
- .

[3]

- (c) Find the optimal transport map between
- f
- and
- g
- for the quadratic cost, and give a proof of why it is optimal.

[4]

4. Let $d \in \mathbb{N}$ and let $\Omega \subset \mathbb{R}^d$ be a compact set and let $\mu \in \mathcal{P}(\Omega)$ be a probability measure. Let $\nu = \delta_{\vec{x}_0}$ with $\vec{x}_0 \in \Omega$ be the Dirac delta mass concentrated at \vec{x}_0 . Let $c : \Omega \times \Omega \rightarrow \mathbb{R}$ be a continuous function and let

$$\Pi(\mu, \nu) := \{\gamma \in \mathcal{P}(\Omega \times \Omega) : (p^x)_\# \gamma = \mu, (p^y)_\# \gamma = \nu\}$$

denote the usual set of transport plans between μ and ν .

- (a) Show that the optimal transport problem from μ to ν

$$\mathcal{T}_c(\mu, \nu) := \inf \left\{ \iint_{\Omega \times \Omega} c(\vec{x}, \vec{y}) d\gamma(\vec{x}, \vec{y}) : \gamma \in \Pi(\mu, \nu) \right\},$$

has a unique solution γ_{opt} , which is concentrated on the graph of a map $T : \Omega \rightarrow \Omega$, i.e. $\gamma_{opt} = (\text{id}, T)_\# \mu$. Find this map. Here $\text{id} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ stands for the identity map, i.e. $\text{id}(\vec{x}) = \vec{x}$. [4]

- (b) Consider the optimal transport problem from ν to μ , i.e.

$$\mathcal{T}_c(\nu, \mu) := \inf \left\{ \iint_{\Omega \times \Omega} c(\vec{x}, \vec{y}) d\gamma(\vec{x}, \vec{y}) : \gamma \in \Pi(\nu, \mu) \right\}.$$

Show that this also has a unique solution, but this cannot be induced by a map, unless $\text{spt}(\mu)$ is a singleton. [3]

- (c) Now let $\mu := \frac{1}{4}\delta_{\vec{y}_1} + \frac{3}{4}\delta_{\vec{y}_2}$, where $\vec{y}_1, \vec{y}_2 \in \Omega$ are distinct points. Compute the optimal values $\mathcal{T}_c(\mu, \nu)$ and $\mathcal{T}_c(\nu, \mu)$. [3]

SECTION B

5. Consider the space of convergent sequences

$$c = \{x = (x_n)_{n \in \mathbb{N}} \subset \mathbb{C} : \lim_{n \rightarrow \infty} x_n \text{ exists}\}.$$

(a) Prove that c is not dense in $\ell^\infty(\mathbb{N})$. [Hint: Show that $y = (y_n)_{n \in \mathbb{N}} \in \ell^\infty(\mathbb{N})$ with $y_n = (-1)^n$ is not approximated by elements in c .] [6]

(b) Using that $\ell^\infty(\mathbb{N})$ can be identified with the dual space of $\ell^1(\mathbb{N})$ (which you can use without proof), show that c is weak-* dense in $\ell^\infty(\mathbb{N})$, i.e. for every $x \in \ell^\infty(\mathbb{N})$ there exists $(x^{(N)})_{N \in \mathbb{N}} \subset c$ with $x^{(N)} \xrightarrow{w^*} x$. [9]

6. (a) State the Closed Graph Theorem. [3]

(b) Let $T : \mathcal{D}(T) \subset \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$ be the linear operator defined by

$$(Tx)_n = nx_n, \quad \mathcal{D}(T) = \left\{ x = (x_n)_{n \in \mathbb{N}} \subset \mathbb{C} : \sum_{n \in \mathbb{N}} |nx_n|^2 < \infty \right\}.$$

Show that T is bijective and its inverse $T^{-1} : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$ is bounded. [5]

(c) Show that the operator T in Question 6.b is closed but not bounded. Why is this not a contradiction to the Closed Graph Theorem? [7]

7. For $d \in \mathbb{N}$ and $r > 0$, let $\alpha_d(r)$ stand for the d -dimensional volume of the d -dimensional ball with radius r . We denote the ball centred at the origin with radius r by

$$B_r := \{\vec{x} \in \mathbb{R}^d : |\vec{x}| \leq r\}.$$

Let $\mu_0 = \frac{1}{\alpha_d(1)} \mathcal{L}^d|_{B_1}$ be the normalised d -dimensional Lebesgue measure restricted to B_1 . Furthermore, define $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$ as $T(\vec{x}) = 2\vec{x}$ and for $t \in [0, 1]$ define

$$\mu_t := ((1-t)\text{id} + tT)_\# \mu_0,$$

where $\text{id} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ stands for the identity map, i.e. $\text{id}(\vec{x}) = \vec{x}$.

- (a) Show that the map $T_t := (1-t)\text{id} + tT$ is invertible for all $t \in [0, 1]$ and compute $(T_t)^{-1}$. [2]
- (b) Determine the density of the measure μ_t , for all $t \in (0, 1]$. [Hint: use the Monge–Ampère equation.] [4]
- (c) Find the optimal transport map in the definition of $W_2(\mu_0, \mu_1)$, which is the 2-Wasserstein distance between μ_0 and μ_1 . [3]
- (d) Define the vector field $\vec{v} : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ as

$$\vec{v}(t, \cdot) = (T - \text{id}) \circ (T_t)^{-1}.$$

Show that the pair $(\mu_t, \vec{v}(t, \cdot))_{t \in [0, 1]}$ is a distributional solution for the continuity equation

$$\partial_t \mu_t + \text{div}(\vec{v}(t, \cdot) \mu_t) = 0$$

on $(0, 1) \times \mathbb{R}^d$. [3]

- (e) Find a constant speed W_2 -geodesic between μ_0 and μ_1 , where W_2 stands for the 2-Wasserstein distance. [3]

8. Let $d \in \mathbb{N}$, $\varepsilon \geq 0$ and let $V : \mathbb{R}^d \rightarrow \mathbb{R}$ be a convex potential function of at most quadratic growth at infinity. Consider the minimisation problem

$$\min \left\{ \int_{\mathbb{R}^d} V(\vec{x}) d\rho(\vec{x}) + \varepsilon \int_{\mathbb{R}^d} \rho(\vec{x}) \log \rho(\vec{x}) d\vec{x} : \rho \in \mathcal{P}_2(\mathbb{R}^d) \right\}.$$

Here, $\mathcal{P}_2(\mathbb{R}^d)$ stands for the set of Borel probability measures supported on \mathbb{R}^d with finite second moment. The logarithmic entropy functional is defined to take the value $+\infty$ for any measure, which is not absolutely continuous with respect to the d -dimensional Lebesgue measure. Throughout this question we assume that this variational problem has a solution, for all $\varepsilon \geq 0$. We will denote this minimiser by ρ_ε .

- (a) Show that the minimiser ρ_ε is unique, for all $\varepsilon > 0$. [2]
- (b) Write down the first order optimality condition satisfied by the minimiser, and give a formula for the minimiser. [5]
- (c) Determine the unique minimiser ρ_0 when $V(\vec{x}) = |\vec{x}|^2$ and $\varepsilon = 0$. [2]
- (d) In the case of $V(\vec{x}) := |\vec{x}|^2$, show that ρ_ε converges in the sense of distributions to ρ_0 (the minimiser obtained in Question 8.c) when $\varepsilon \rightarrow 0$. [*Hint*: you may use without proof that $\int_{\mathbb{R}^d} e^{-|\vec{x}|^2} d\vec{x} = \pi^{d/2}$.] [6]
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