Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH43820-WE01

Title:
Superstrings V

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

SECTION A

1. Recall the Polyakov action for a bosonic string in conformal gauge:

$$S = -\frac{T}{2}\int d^2\sigma\, \eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu},$$

where $\eta_{\alpha\beta}$ and $\eta_{\mu\nu}$ are the Minkowski metric on the worldsheet and target space, respectively. Consider an open string with Neumann boundary conditions.

(a) Find the equations of motion for the worldsheet fields X^μ by varying the action and keeping track of boundary conditions. **[4]**

(b) Recall that the mode expansion for an open string with Neumann boundary conditions is

$$X^\mu(\tau,\sigma) = x_0^\mu + 2\alpha' p^\mu \tau + 2i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu \cos(n\sigma)\, e^{-in\tau}.$$

Moreover, the angular momentum is given by

$$J^{\mu\nu} = T\int_0^\pi d\sigma\, \left(X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu\right).$$

Show that in the classical theory

$$J^{\mu\nu} = x_0^\mu p^\nu - x_0^\nu p^\mu - i\sum_{n=1}^{\infty}\frac{1}{n}\left(\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu\right).$$

Hint: The following formula may be useful:

$$\int_0^\pi d\sigma\, \cos(n\sigma)\cos(m\sigma) = \frac{\pi}{2}\left(\delta_{n,m} + \delta_{n,-m}\right).$$ **[6]**

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2. Consider the scattering of a graviton with momentum p_1^μ and two tachyons with momenta p_2^μ and p_3^μ in closed bosonic string theory.

(a) Show that $p_1 \cdot p_2 = p_1 \cdot p_3 = 0$ and $p_2 \cdot p_3 = -4/\alpha'$. **Hint**: Use momentum conservation $\sum_{i=1}^3 p_i^\mu = 0$. **[5]**

(b) The scattering amplitude is described by the following worldsheet integral:

$$\mathcal{A}_3 \propto \int \frac{\prod_{i=1}^3 d^2 z_i}{\mathrm{Vol}\,(\mathrm{SL}(2,\mathbb{C}))}\frac{1}{|z_{23}|^4}\epsilon_{\mu\nu}\left(\frac{p_2^\mu}{z_{12}} + \frac{p_3^\mu}{z_{13}}\right)\left(\frac{p_2^\nu}{\bar{z}_{12}} + \frac{p_3^\nu}{\bar{z}_{13}}\right),$$

where $z_{ij} = z_i - z_j$ and $\epsilon_{\mu\nu}$ is the polarisation of the graviton. Show that

$$\mathcal{A}_3 \propto \epsilon_{\mu\nu}\left(p_2^\mu - p_3^\mu\right)\left(p_2^\nu - p_3^\nu\right).$$

Hint: Use the $\mathrm{SL}(2,\mathbb{C})$ symmetry and recall that $\epsilon_{\mu\nu}p_1^\mu = 0$. **[5]**

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3. Consider the two 2×2 matrices ρ^a with $a = 1, 2$,

$$\rho^1 = \sigma_1 \,, \qquad \rho^2 = \sigma_3 \,,$$

where the Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

(a) Show that the ρ^a matrices satisfy the two-dimensional Euclidean Clifford algebra

$$\{\rho^a, \rho^b\} = 2\delta^{ab}\mathbb{1}_2 \,,$$

where δ^{ab} denotes the two-dimensional Kronecker delta symbol and $\mathbb{1}_2$ is the 2×2 identity matrix. **[5]**

(b) Define the matrices

$$S^{ab} = \frac{1}{4}[\rho^a, \rho^b] \,,$$

and compute the transformation

$$\exp(\theta_{ab} S^{ab})\Psi$$

of the two-dimensional Euclidean spinor $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, where $\theta_{ab} = -\theta_{ba} \in \mathbb{R}$.

Hint: Remember the Taylor series $\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. You may also wish to use $\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$ and $\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$. **[5]**

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4. Consider a system with one set of fermionic creation/annihilation operators, $\hat{b}^\dagger, \hat{b}$, with canonical anti-commutation relations

$$\{\hat{b}, \hat{b}^\dagger\} = 1 \,, \qquad \{\hat{b}, \hat{b}\} = \{\hat{b}^\dagger, \hat{b}^\dagger\} = 0 \,.$$

Define the operator

$$\hat{B} = \alpha\hat{b} + \beta\hat{b}^\dagger$$

with $\alpha, \beta \in \mathbb{R}$. Compute $\hat{B}^n$ for all $n \in \mathbb{N}$ with $n \geq 2$. **[10]**

SECTION B

5. Recall that the worldsheet fields in closed bosonic string theory have the following operator product expansion:

$$X(z)\,X(w) = -\frac{\alpha'}{2}\ln(z-w) + ..., \quad \tilde{X}(\bar{z})\,\tilde{X}(\bar{w}) = -\frac{\alpha'}{2}\ln(\bar{z}-\bar{w}) + ...$$

where we take the target space to be one-dimensional, we write $X(z,\bar{z}) = X(z) + \tilde{X}(\bar{z})$, and ... denotes non-singular terms. Note that in the lectures we denoted $\tilde{X}(\bar{z})$ as $X(\bar{z})$. In this case the components of the stress tensor are given by

$$T(z) = -\frac{1}{\alpha'} : \partial X(z)\partial X(z) :, \quad \tilde{T}(\bar{z}) = -\frac{1}{\alpha'} : \bar{\partial}\tilde{X}(\bar{z})\,\bar{\partial}\tilde{X}(\bar{z}) : .$$

(a) State the definition of a conformal primary operator of weight $(h, \tilde{h})$. **[3]**

(b) Compute the singular terms in the operator product expansion of $T(z)$ with $\Phi(w) = \partial^m X(w)$, where m is any non-negative integer. **[6]**

(c) Compute the singular terms in the operator product expansion of $T(z)$ with $\Phi(w) =: X(w)^m :$, where m is any positive integer. **[6]**

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6. The Polyakov action for a closed bosonic string in a general spactime background is

$$S = -\frac{T}{2}\int d^2\sigma\,\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X),$$

where $h_{\alpha\beta}$ is the worldsheet metric, h is its determinant, and $G_{\mu\nu}$ is target space metric, which can be a general function of the target space coordinates X^μ.

(a) Show that in conformal gauge the Virasoro constraints are given by

$$G_{\mu\nu}\left(\partial_\tau X^\mu \partial_\tau X^\nu + \partial_\sigma X^\mu \partial_\sigma X^\nu\right) = G_{\mu\nu}\partial_\tau X^\mu \partial_\sigma X^\nu = 0.$$

Hint: Recall that $\delta\sqrt{-h} = -\frac{1}{2}\sqrt{-h}h_{\alpha\beta}\delta h^{\alpha\beta}$ under a general variation of the worldsheet metric. **[4]**

(b) Suppose the string moves in the following background:

$$G_{\mu\nu}dX^\mu dX^\nu = R^2\left(-dt^2\,\cosh^2\rho + d\rho^2 + \sinh^2\rho\,d\phi^2\right),$$

which describes three-dimensional Anti-de Sitter space with radius R. Moreover suppose that $t = \varepsilon\tau$, $\phi = \omega t$, and ρ is a general function of σ. Show that in conformal gauge

$$(\partial_\sigma\rho)^2 = \varepsilon^2\left(\cosh^2\rho - \omega^2\sinh^2\rho\right).$$

What is the maximum value of ρ that the string can have assuming $\omega^2 \geq 1$? **[6]**

(c) Using the parameterisation and metric in part (b), show that the following quantities are conserved in conformal gauge:

$$E = \frac{R^2}{2\pi\alpha'}\varepsilon\int_0^{2\pi} d\sigma\,\cosh^2\rho, \quad J = \frac{R^2}{2\pi\alpha'}\varepsilon\omega\int_0^{2\pi} d\sigma\,\sinh^2\rho.$$

What is their physical interpretation? **[5]**

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7. Consider the quantum mechanical Hamiltonian

$$\hat{H} = \hat{a}^\dagger \hat{a} + \hat{f}_1^\dagger \hat{f}_1 + \hat{f}_2^\dagger \hat{f}_2 \,,$$

where $a^\dagger, a$ are canonical bosonic creation/annihilation operators and $f_i^\dagger, f_i$ with $i = 1, 2$ are canonical fermionic creation/annihilation operators, i.e.

$$[\hat{a}, \hat{a}^\dagger] = 1 \,, \qquad [\hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0 \,,$$
$$\{\hat{f}_i, \hat{f}_j^\dagger\} = \delta_{ij} \,, \qquad \{\hat{f}_i, \hat{f}_j\} = \{\hat{f}_i^\dagger, \hat{f}_j^\dagger\} = 0 \,, \qquad i, j = 1, 2 \,.$$

Bosonic and fermionic operators commute with each other:

$$[\hat{a}, \hat{f}_i] = [\hat{a}^\dagger, \hat{f}_i] = [\hat{a}, \hat{f}_i^\dagger] = [\hat{a}^\dagger, \hat{f}_i^\dagger] = 0 \,, \qquad i = 1, 2 \,.$$

The Fock vacuum $|0\rangle$ is the unique normalised state annihilated by $\hat{a}$ and $\hat{f}_i$,

$$\hat{a}\,|0\rangle = \hat{f}_i\,|0\rangle = 0 \,, \qquad i = 1, 2 \,.$$

(a) Find the eigenstates of the Hamiltonian and their corresponding eigenvalues (do not worry about obtaining normalised states). **[4]**

(b) Let $(-1)^{\hat{F}}$ denote the total fermion number operator such that it anti-commutes with all fermionic operators and commutes with all bosonic ones, i.e.

$$\{(-1)^{\hat{F}}, \hat{f}_i\} = \{(-1)^{\hat{F}}, \hat{f}_i^\dagger\} = 0 \,, \qquad i = 1, 2 \,,$$
$$[(-1)^{\hat{F}}, \hat{a}] = [(-1)^{\hat{F}}, \hat{a}^\dagger] = 0 \,.$$

If we assume that the vacuum $|0\rangle$ is bosonic, i.e. $(-1)^{\hat{F}}\,|0\rangle = +\,|0\rangle$, find all the bosonic eigenstates of the Hamiltonian, i.e. all the eigenstates $|\psi\rangle$ such that $(-1)^{\hat{F}}\,|\psi\rangle = +\,|\psi\rangle$. **[3]**

(c) In the same setup as above assume now that the vacuum $|0\rangle$ is fermionic, i.e. $(-1)^{\hat{F}}\,|0\rangle = -\,|0\rangle$, find all the bosonic eigenstates of the Hamiltonian, i.e. all the eigenstates $|\psi\rangle$ such that $(-1)^{\hat{F}}\,|\psi\rangle = +\,|\psi\rangle$. **[3]**

(d) Let $x \in \mathbb{R}$ be a real number with $|x| < 1$. Compute

$$\mathrm{Tr}\, x^{\hat{H}} = \sum_j \frac{\langle \psi_j |\, x^{\hat{H}}\, | \psi_j \rangle}{\langle \psi_j \,| \psi_j \rangle} \,,$$

where the sum runs over an orthogonal basis $\{|\psi_j\rangle\}$ of states.

Hint: Which basis should you choose as to make the calculation of $\langle \psi_j |\, x^{\hat{H}}\, | \psi_j \rangle$ the easiest possible? There is no need to compute $\langle \psi_j \,| \psi_j \rangle$. **[5]**

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8. The RNS gauge-fixed Polyakov action is given by

$$S_P = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\,\eta^{ab}\left(\partial_a X^\mu \partial_b X^\nu + i\bar{\Psi}^\mu \rho_a \partial_b \Psi^\nu\right)\eta_{\mu\nu}\,.$$

The worldsheet coordinates are (τ, σ) and $X^\mu(\tau, \sigma)$ are D worldsheet bosonic fields, while $\Psi^\mu(\tau, \sigma)$ are D worldsheet Majorana fermions. The 2-dimensional Dirac matrices are

$$\rho^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \qquad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,.$$

(a) Compute the equations of motions for $X^\mu(\tau, \sigma)$ and $\Psi^\mu(\tau, \sigma)$. You may discard boundary terms. **[4]**

(b) Given the supercurrent definition

$$J_a = \frac{1}{2}\rho^b \rho_a \Psi^\mu \partial_b X_\mu\,,$$

compute $J_\pm$ in terms of the spinor components $\Psi^\mu = \begin{pmatrix} \psi_+^\mu \\ \psi_-^\mu \end{pmatrix}$. **[3]**

(c) Let us fix the light-cone gauge condition

$$X^+ = \frac{X^0 + X^1}{\sqrt{2}} = p^+\tau\,.$$

Use the supersymmetry transformation

$$\delta\Psi^\mu = \rho^a \partial_a X^\mu \epsilon\,,$$

to compute the supersymmetry variation $\delta\Psi^+$ with

$$\Psi^+ = \frac{\Psi^0 + \Psi^1}{\sqrt{2}}\,.$$

[4]

(d) Assume that the supersymmetry parameters $\epsilon = \begin{pmatrix} \epsilon_+ \\ \epsilon_- \end{pmatrix}$ satisfy $\partial_\pm \epsilon_\pm = 0$. Combine the fermionic equations of motions you found in part (a) with the supersymmetry variation $\delta\Psi^+$ computed in part (c) to show that it is possible to choose a local supersymmetry parameter ϵ such that the fermionic light-cone gauge condition

$$\Psi^+ + \delta\Psi^+ = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\,,$$

holds. **[4]**
