EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH43920-WE01

Title:
Topics in Combinatorics V

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

SECTION A

1. Denote by $e_k(n)$ the number of Young diagrams $\lambda \vdash n$ with exactly k rows, that have **all rows of even length**, and denote by $o_k(n)$ the number of Young diagrams $\lambda \vdash n$ with exactly k rows, that have **all rows of odd length**.

(a) Show that $e_k(n) = o_k(n-k)$ for all $n > k > 0$. **[5]**

(b) Is it true that $o_k(n) = e_k(n-k)$ for all $n > k > 0$? If so, prove the identity; if not, find a correct expression for $o_k(n)$ only in terms of $e_\ell(m)$ for integers $\ell \leq k$ and $m \leq n$. **[5]**

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2. (a) Let $w_A = 4\,7\,8\,3\,1\,5\,2\,9\,6 \in S_9$. Apply the Robinson–Schensted–Knuth (RSK) algorithm to compute the insertion and recording tableaux P and Q. **[4]**

(b) Let R be a standard Young tableau of shape $\lambda = (3,3,2,1) \vdash 9$, where

$$R = \begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 2 & 6 & 7 \\ \hline 3 & 9 \\ \cline{1-2} 8 \\ \cline{1-1}\end{array} .$$

Find $w_B \in S_9$ that satisfies $(w_B)^2 = \mathrm{Id}$ and whose RSK insertion tableau equals R. **[4]**

(c) Find another $w_C \in S_9$, not equal to w_B, whose RSK insertion tableau also equals R. **[2]**

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3. Let $\Gamma = \langle s_1, s_2, s_3 \mid s_i^2, (s_1 s_2)^4, (s_2 s_3)^3, (s_1 s_3)^4, s_2^7 \rangle$.

(a) Define $\Gamma' = \langle s_1, s_3 \mid \text{all relations in } \Gamma \text{ not containing } s_2 \rangle$. Find the order of Γ'. **[5]**

(b) Show that the subgroup of Γ generated by s_1 and s_3 has order 2. **[5]**

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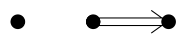
4. (a) Let P be the root poset of a root system Δ with disconnected Dynkin diagram shown below.



Draw the Hasse diagram of P. **[5]**

(b) Draw the Hasse diagram of the poset of order ideals of P. Identify join-irreducible elements. **[5]**

SECTION B

5. A Dyck path from $(0,0)$ to $(2n,0)$ has a *mountain* if it passes through the points $(2k-2,0),(2k,2)$ and $(2k+2,0)$ for some k satisfying $0<k<n$. Denote by G_n the number of *mountain-free* Dyck paths of length $2n$, i.e., Dyck paths of length $2n$ that do not have a mountain. Note that $G_0=1$ (the trivial Dyck path does not have a mountain).

(a) Argue carefully that $(G_n)_{n\geq 0}$ satisfies a recurrence of the form

$$C_n = G_n + \sum_{k=1}^{n-1} G_{a(k,n)} C_{b(k,n)}$$

where $a(k,n)$ and $b(k,n)$ are integer expressions in k and n that should be determined, and $(C_n)_{n\geq 0}$ is the sequence of Catalan numbers. **[6]**

(b) Compute the generating function $G(x)$ of the sequence $(G_n)_{n\geq 0}$ and show that

$$G(x) = \frac{\alpha}{\beta + \gamma x^2 + \sqrt{1-4x}}$$

where α, β and γ are constants to be determined. **[9]**

Hint: You may use without proof that the generating function $C(x)$ of the Catalan numbers is

$$C(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

6. Given a permutation $w \in S_n$, let $\mathrm{cyc}(w)$ denote the number of cycles of w, and let $f_{\mathrm{cyc}}(x) := \sum_{w \in S_n} x^{\mathrm{cyc}(w)}$ be the generating function of cyc.

(a) For $n = 3$, calculate $\mathrm{cyc}(w)$ for each permutation $w \in S_3$, and show that

$$f_{\mathrm{cyc}}(x) = (x + a_1)(x + a_2)(x + a_3),$$

where a_1, a_2 and a_3 are integers that you should determine. **[3]**

(b) For integers $n \geq 1$ and $k \geq 0$, define $c(n, k) := |\{w \in S_n : \mathrm{cyc}(w) = k\}|$ to be the number of permutations of S_n that have k cycles. Prove that

$$c(n + 1, k) = c(n, k - 1) + nc(n, k)$$

for all $n \geq 1$ and $1 \leq k \leq n + 1$. **[4]**

(c) Using part (b) or otherwise, state and prove a formula for $f_{\mathrm{cyc}}(x)$ that holds for general $n \in \mathbb{N}$. **[4]**

(d) Recall that, for $w = w_1 w_2 \ldots w_n \in S_n$ written in 1-line notation, w_i is a *record* of w if $w_j < w_i$ for all $j < i$. Denote by $\mathrm{rec}(w)$ the number of records of w, and let $f_{\mathrm{rec}}(x)$ be the generating function of rec. By considering an appropriate bijection from S_n to itself, prove that $f_{\mathrm{rec}}(x) = f_{\mathrm{cyc}}(x)$ for all $n \in \mathbb{N}$. **[4]**

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7. Let (G, S) be a Coxeter system, R be the set of reflections of (G, S), and l be the length function. Define a partial order on (G, S) as follows: for $g_1, g_2 \in G$, we say that $g_1 \leq g_2$ if there exist $t_1, \ldots, t_k \in R$ such that $g_2 = g_1 t_1 \ldots t_k$ and $l(g_1 t_1 \ldots t_i t_{i+1}) > l(g_1 t_1 \ldots t_i)$ for every $i = 1, \ldots, k - 1$ (and $l(g_1 t_1) > l(g_1)$). The partial order $\leq$ is called the *Bruhat order*.

(a) Draw the Hasse diagram of the Bruhat order on the Weyl group of type $A_2 = I_2(3)$ (justify your answer). **[5]**

(b) Let (G, S) be a Coxeter system of type A_3. Give an example of $g_1, g_2 \in G$ with $l(g_1) \leq l(g_2)$ but g_1 being incomparable with g_2 (justify your answer). **[5]**

(c) Let $G = I_2(n)$ be a dihedral group. Show that $g_1 \leq g_2$ if and only if $l(g_1) \leq l(g_2)$. **[5]**

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8. Let Δ be the root system of type C_4. Let Δ_l and Δ_s be the sets of long and short roots of Δ respectively.

(a) Show that Δ_l and Δ_s are root systems and find their types. **[5]**

(b) Compute the Coxeter number of Δ_s. **[5]**

(c) Find the exponents of the Weyl group of Δ_s. **[5]**
