EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH4411-WE01

Title:
Advanced Mathematical Biology IV

Time:	3 hours	
Additional Material provided:	Formula sheet	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

SECTION A

1. Consider the stochastic process X satisfying the following computational definition of an SDE:

$$X(t+\Delta t) = X(t) + (\mu - \lambda X(t)^2)\,\Delta t + \sigma\sqrt{\Delta t}\,\xi, \qquad t \geq 0,$$

where $\mu, \lambda, \sigma > 0$ are constants, $\xi \sim N(0,1)$ are independent for each time step, and $X(0) = x_0$. Let $M(t) = \mathbb{E}[X(t)]$ and $S(t) = \mathbb{E}[X(t)^2]$ denote the first and second moments of $X(t)$, respectively.

(a) Using the computational definition above, derive ordinary differential equations (ODEs) satisfied by $M(t)$ and $S(t)$ by computing $M(t+\Delta t)$ and $S(t+\Delta t)$ in terms of $M(t)$ and $S(t)$, and taking the limit $\Delta t \downarrow 0$. **[4]**

(b) Comment briefly on whether the ODE system you derived in part (a) forms a closed system, i.e., can you solve for $M(t)$ and $S(t)$ without computing higher moments? Explain your reasoning. **[2]**

(c) Using the general stationary distribution formula for SDEs, compute the stationary distribution $p_s(x)$ up to a normalising constant. **[4]**

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2. A single chemical species A evolves in a well-mixed container of volume ν according to the reaction dynamics

$$2A \xrightarrow{k_1} 3A, \qquad A \xrightarrow{k_2} \emptyset, \qquad \emptyset \xrightarrow{k_3} A,$$

with rate constants $k_1, k_2, k_3 > 0$. Let $A(t)$ denote the number of A molecules at time t, and define $P_n(t) = \mathbb{P}[A(t) = n]$ for $n \geq 0$.

(a) Use the Law of Mass Action to derive a deterministic ODE for the concentration $a(t) = A(t)/\nu$. Analyse the steady states of this ODE and determine conditions on k_1, k_2, k_3 under which the mass action system is monostable or bistable. **[4]**

(b) Write down the chemical master equations for $P_n(t)$. **[2]**

(c) Define the probability generating function

$$G(x,t) = \sum_{n=0}^{\infty} x^n P_n(t), \quad x \in (-1,1).$$

Derive an evolution equation for $G(x,t)$ using the chemical master equations in part (b). **[4]**

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3. Consider a Bingham plastic fluid flowing in a concentric cylinder rheometer. The inner cylinder of radius R_1 is rotated at a constant angular velocity Ω, while the outer cylinder of radius R_2 is stationary. Both cylinders have the same length $L \gg R_2$, and the gap between them is narrow, i.e., $R_2/R_1 \approx 1$. The flow is considered to be incompressible, steady, laminar, and axisymmetric. No pressure gradients arise, and the effect of gravity is negligible. You may assume that end effects are also negligible.

(a) Derive the azimuthal velocity u_θ profile in the following cases: (i) $|\tau| < \tau_0$, and (ii) $|\tau| > \tau_0$, where τ_0 denotes the yield stress. **[5]**

(b) Determine a criterion for the onset of fluid flow. **[2]**

(c) Assume that the narrow-gap assumption does not hold true. Providing brief explanations, make qualitative plots of the azimuthal velocity profile in the following cases:

(i) $|\tau| < \tau_0$ for $r \in [R_1, R_2]$,

(ii) $|\tau| > \tau_0$ for $r \in [R_1, R']$ and $|\tau| < \tau_0$ for $r \in [R', R_2]$, where R' is the radial distance at which $|\tau| = \tau_0$, and

(iii) $|\tau| > \tau_0$ for $r \in [R_1, R_2]$. **[3]**

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4. A watery solution is placed in a parallel disk rheometer to measure its rheological properties. The watery solution is modelled as an incompressible, Newtonian fluid. It flows in the gap of thickness H between the two parallel disks of radius R. The upper disk rotates with a constant angular velocity Ω, while the lower disk is fixed. The flow is considered to be steady, laminar, and axisymmetric. No pressure gradients arise, and the effect of gravity is negligible. Consider a cylindrical coordinate system (r, θ, x), with $r \in [0, R]$, $\theta \in [0, 2\pi]$, and $x \in [0, H]$.

(a) Derive the azimuthal velocity u_θ profile, assuming that it varies linearly with the radial distance r and is also a function of the axial distance x from the lower disk. **[6]**

(b) Determine an expression for the solution viscosity μ, given experimental measurements of torque magnitude $\mathcal{M}$ required to maintain the upper disk rotating at a constant Ω. **[4]**

SECTION B

5. Consider a chemical reaction system involving three species A, B, and C in a well-mixed container of volume ν. The reaction dynamics are given by:

$$\emptyset \xrightarrow{k_1} C, \qquad C \xrightarrow{k_2} B \xrightarrow{k_3} \emptyset, \qquad A + C \xrightarrow{k_4} A, \qquad \emptyset \xrightarrow{k_5} A \xrightarrow{k_6} \emptyset.$$

(a) Using the Law of Mass Action, write down the system of ordinary differential equations (ODEs) describing the concentrations $a(t), b(t), c(t)$ of the three species, and find the steady-state concentrations a^*, b^*, c^* in terms of the rate constants $k_1, \dots, k_6$. **[4]**

(b) Suppose the production rate of A, k_5, fluctuates between two values, k_5^{high} and k_5^{low}, on a slow timescale.

Based on the steady-state expressions from part (a), determine how the steady-state concentration of B depends on k_5. Does the deterministic model predict that B is more sensitive, less sensitive, or equally sensitive to changes in k_5 compared to A? **[2]**

(c) Consider the regime where A fluctuates stochastically, but C remains at very low copy numbers, i.e. system parameters are such that $c^* \ll 1$.

Briefly explain the phenomenon of stochastic focusing, including the necessary ingredients for it be observed in a given model, and comment on whether it is possible in the present model. **[3]**

(d) State an example of a biological system or process where stochastic focusing has been suggested to explain the observed dynamics. **[1]**

(e) Assuming the dynamics of C are fast relative to A, explain why the conditional stationary mean number of molecules of C given n_A molecules of A can be approximated well by the following formula:

$$\mathbb{E}[C(t) \mid A(t) = n_A] = \frac{k_1 \nu}{k_2 + k_4 n_A / \nu}. \tag{5.1}$$

Hence, use the expression (5.1) to derive a summation formula for the (unconditional) mean of C, $\mathbb{E}[C(t)]$. **[5]**

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6. Consider a stochastic reaction–diffusion system involving a single species A in a one-dimensional domain $[0, L]$. The domain is divided into two well-mixed compartments of equal width $h = L/2$, denoted Compartment 1 and Compartment 2. We assume that the effective "volume" of each compartment is $\nu = 1$. The dynamics are defined as follows:

- Molecules of A can diffuse between Compartment 1 and Compartment 2 with hopping rate $d > 0$.
- In Compartment 1 only, molecules are produced at a constant rate k_1 and degraded at a rate $k_2 n_1$, where n_1 is the number of molecules in Compartment 1.
- There are no reactions in Compartment 2, only diffusion.

(a) Write down the chemical master equations for the joint probability $P(n_1, n_2, t)$, where n_i is the number of molecules in compartment i. **[3]**

(b) Consider the limit where diffusion is very fast relative to the reaction rates ($d \gg k_1, k_2$) so that diffusion reactions dominate the dynamics.

Argue that the average number of molecules in Compartment 1 is $N/2$ and show that the total number of molecules $N = n_1 + n_2$ evolves approximately according to a one-dimensional production–degradation process, specifying the effective production and degradation propensities for N. **[4]**

(c) Using the effective one-dimensional process for N derived in part (b), write down the corresponding Chemical Fokker–Planck equation for the probability density $P(x, t)$, where x represents the continuous approximation of the total molecule count N. **[3]**

(d) The system is initialized with $N = 0$ molecules. We wish to calculate the mean time T required for the total population to reach a given threshold value $N_c > 0$.

Using the Chemical Fokker–Planck approximation, write down an expression for the mean hitting time T. [**You do NOT need to solve the integrals involved explicitly.**] **[3]**

(e) Explain qualitatively how increasing the hopping rate d would affect the mean hitting time T if the assumption $d \gg k_1, k_2$ were relaxed. **[2]**

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7. Blood flows inside a vessel, which can be modelled as a cylindrical tube of radius R and length L. Blood enters the vessel, where $x = 0$, at a constant velocity U_0 along the x-direction. At the vessel's outlet, where $x = L$, the velocity profile of blood is given by

$$u_x(r) = u_{\max}\left[1 - \left(\frac{r}{R}\right)^3\right].$$

The flow is considered to be incompressible, steady, and laminar. Consider a cylindrical coordinate system (r, θ, x), with $r \in [0, R]$, $\theta \in [0, 2\pi]$, and $x \in [0, L]$.

(a) By making use of the mass conservation law, show that the maximum velocity satisfies the following: $u_{\max} = 5U_0/3$. **[3]**

(b) Assume that the pressure is uniform and equal to constants p_0 and p_L at the vessel's entrance and end, respectively. Determine the pressure force. **[3]**

(c) By making use of the momentum conservation law, determine the total force exerted on the blood. Express your answer in terms of the blood density ρ, the velocity U_0, and the vessel radius R. **[5]**

(d) A catheter is inserted inside the vessel, without causing any disturbance to the velocity profiles at the vessel's entrance and end. Estimate the force acting on the catheter by the blood. You may assume that: (i) shear stresses are negligible, and (ii) the effect of gravity on blood flow is also negligible. **[4]**

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8. Respiratory mucus is placed in a sliding plate rheometer to measure its rheological properties. The mucus flows between two parallel plates, separated by a gap of thickness H. The lower plate is stationary, while the upper plate moves at a constant velocity U along the x-direction. The flow is assumed to be incompressible, steady, laminar, and fully developed. Edge effects are negligible. No pressure gradients arise, and the effect of gravity is negligible. Consider a Cartesian coordinate system (x, y, z), where $x \in [0, L]$, $y \in [0, H]$, and $z \in [0, W]$, with L and W being, respectively, the length and width of the plates.

(a) Respiratory mucus is modelled as a Herschel–Bulkley fluid with a constitutive equation given by

$$\tau = \left[\operatorname{sign}(\dot{\gamma})\ \frac{\tau_0}{|\dot{\gamma}|} + \kappa|\dot{\gamma}|^{n-1}\right]\dot{\gamma}, \text{ for } |\tau| > \tau_0, \text{ and}$$
$$\dot{\gamma} = 0, \text{ for } |\tau| < \tau_0.$$

Explain the physical meaning of the constitutive equation parameters τ_0, κ, and n. **[3]**

(b) Show that the shear stress τ_{yx} distribution is uniform across the sliding plate rheometer, that is $\tau_{yx}(y) = c_1$, where $c_1 > 0$ is a constant. **[3]**

(c) Derive the velocity profile of mucus in a sliding plate rheometer. You may assume that $|\tau_{yx}| > \tau_0$. **[5]**

(d) Find a relationship between the experimentally measured force, $\mathbf{F}$, required to move the upper plate at a constant velocity U and the Herschel–Bulkley model parameters, namely τ_0, κ, and n. Explain how these model parameters can be experimentally determined. **[4]**
