Formula Sheet for MATH 44120

Geophysical and Astrophysical Fluids IV

1 Some vector identities

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = (\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C} \tag{F0}$$

$$\nabla\cdot(f\boldsymbol{A}) = (\nabla f)\cdot\boldsymbol{A} + f\nabla\cdot\boldsymbol{A} \tag{F1}$$

$$\nabla\times(f\boldsymbol{A}) = (\nabla f)\times\boldsymbol{A} + f\nabla\times\boldsymbol{A} \tag{F2}$$

$$\nabla\times(\nabla\times\boldsymbol{A}) = \nabla(\nabla\cdot\boldsymbol{A}) - \nabla^2\boldsymbol{A} \tag{F3}$$

$$(\boldsymbol{A}\cdot\nabla)\boldsymbol{A} = \tfrac{1}{2}\nabla|\boldsymbol{A}|^2 - \boldsymbol{A}\times(\nabla\times\boldsymbol{A}) \tag{F4}$$

$$\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{B}\cdot(\nabla\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\nabla\times\boldsymbol{B}) \tag{F5}$$

$$\nabla\times(\boldsymbol{A}\times\boldsymbol{B}) = (\boldsymbol{B}\cdot\nabla)\boldsymbol{A} - (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} + \boldsymbol{A}\nabla\cdot\boldsymbol{B} - \boldsymbol{B}\nabla\cdot\boldsymbol{A} \tag{F6}$$

$$\nabla(\boldsymbol{A}\cdot\boldsymbol{B}) = (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} + (\boldsymbol{B}\cdot\nabla)\boldsymbol{A} + \boldsymbol{A}\times(\nabla\times\boldsymbol{B}) + \boldsymbol{B}\times(\nabla\times\boldsymbol{A}) \tag{F7}$$

2 In cylindrical coordinates (r,θ,z)

$$\nabla f = \boldsymbol{e}_r\partial_r f + \frac{\boldsymbol{e}_\theta}{r}\partial_\theta f + \boldsymbol{e}_z\partial_z f \tag{F8}$$

$$\nabla\cdot\boldsymbol{A} = \frac{1}{r}\partial_r(rA_r) + \frac{1}{r}\partial_\theta A_\theta + \partial_z A_z \tag{F9}$$

$$\nabla\times\boldsymbol{A} = \Big(\frac{1}{r}\partial_\theta A_z - \partial_z A_\theta\Big)\boldsymbol{e}_r + (\partial_z A_r - \partial_r A_z)\boldsymbol{e}_\theta + \frac{1}{r}(\partial_r(rA_\theta) - \partial_\theta A_r)\boldsymbol{e}_z \tag{F10}$$

$$\nabla^2 f = \frac{1}{r}\partial_r(r\,\partial_r f) + \frac{1}{r^2}\partial_{\theta\theta}f + \partial_{zz}f \tag{F11}$$

$$\nabla^2\boldsymbol{A} = \Big(\nabla^2 A_r - \frac{1}{r^2}A_r - \frac{2}{r^2}\partial_\theta A_\theta\Big)\boldsymbol{e}_r + \Big(\nabla^2 A_\theta + \frac{2}{r^2}\partial_\theta A_r - \frac{1}{r^2}A_\theta\Big)\boldsymbol{e}_\theta + \boldsymbol{e}_z\nabla^2 A_z \tag{F12}$$

$$(\boldsymbol{B}\cdot\nabla)\boldsymbol{A} = \boldsymbol{e}_r\Big(\boldsymbol{B}\cdot\nabla A_r - \frac{B_\theta A_\theta}{r}\Big) + \boldsymbol{e}_\theta\Big(\boldsymbol{B}\cdot\nabla A_\theta + \frac{B_\theta A_r}{r}\Big) + \boldsymbol{e}_z\,\boldsymbol{B}\cdot\nabla A_z \tag{F13}$$

3 In spherical coordinates (r, θ, ϕ)

$$\nabla f = \boldsymbol{e}_r \partial_r f + \frac{\boldsymbol{e}_\theta}{r} \partial_\theta f + \frac{\boldsymbol{e}_\phi}{r \sin\theta} \partial_\phi f \tag{F14}$$

$$\nabla\cdot \boldsymbol{A} = \frac{1}{r^2}\partial_r(r^2 A_r) + \frac{1}{r\sin\theta}\partial_\theta(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\partial_\phi A_\phi \tag{F15}$$

$$\nabla\times \boldsymbol{A} = \frac{\boldsymbol{e}_r}{r\sin\theta}(\partial_\theta(A_\phi \sin\theta) - \partial_\phi A_\theta) + \frac{\boldsymbol{e}_\theta}{r}\Big(\frac{1}{\sin\theta}\partial_\phi A_r - \partial_r(rA_\phi)\Big) + \frac{\boldsymbol{e}_\phi}{r}(\partial_r(rA_\theta) - \partial_\theta A_r) \tag{F16}$$

$$\nabla^2 f = \frac{1}{r}\partial_{rr}(rf) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta f) + \frac{1}{r^2\sin^2\theta}\partial_{\phi\phi} f \tag{F17}$$

$$\begin{aligned}\nabla^2 \boldsymbol{A} = &\Big(\nabla^2 A_r - \frac{2}{r^2}A_r - \frac{2}{r^2\sin\theta}\left[\partial_\theta(\sin\theta A_\theta) + \partial_\phi A_\phi\right]\Big)\boldsymbol{e}_r \\ &+ \Big(\nabla^2 A_\theta + \frac{2}{r^2}\partial_\theta A_r - \frac{A_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\partial_\phi A_\phi\Big)\boldsymbol{e}_\theta \\ &+ \Big(\nabla^2 A_\phi + \frac{2}{r^2\sin\theta}\partial_\phi A_r + \frac{2\cos\theta}{r^2\sin^2\theta}\partial_\phi A_\theta - \frac{A_\phi}{r^2\sin^2\theta}\Big)\boldsymbol{e}_\phi\end{aligned} \tag{F18}$$

$$\begin{aligned}(\boldsymbol{B}\cdot\nabla)\boldsymbol{A} = &\,\boldsymbol{e}_r\Big(\boldsymbol{B}\cdot\nabla A_r - \frac{B_\theta A_\theta}{r} - \frac{B_\phi A_\phi}{r}\Big) + \boldsymbol{e}_\theta\Big(\boldsymbol{B}\cdot\nabla A_\theta - \frac{B_\phi A_\phi}{r}\cot\theta + \frac{B_\theta A_r}{r}\Big) \\ &+ \boldsymbol{e}_\phi\Big(\boldsymbol{B}\cdot\nabla A_\phi + \frac{B_\phi A_r}{r} + \frac{B_\phi A_\theta}{r}\cot\theta\Big)\end{aligned} \tag{F19}$$

4 Bessel functions

Bessel functions $u(r) = J_n(r)$ and $u(r) = Y_n(r)$ are solutions to the ODE

$$r^2 u'' + ru' + (r^2 - n^2)u = 0. \tag{F20}$$

Both $J_n(r)$ and $Y_n(r) \to 0$ as $r \to \infty$; $J_n(0) = \delta_{n0}$, and $|Y_n(r)| \to \infty$ as $r \to 0$.

5 Legendre polynomials

When ℓ is an integer, the ODE

$$(1 - s^2)u'' - 2su' + \ell(\ell+1)u = 0, \tag{F21}$$

has polynomial solutions $P_\ell(s)$.

[The other solutions $Q_\ell(s)$ are singular at $s = \pm 1$, and for non-integer ℓ neither set of solutions is well defined at $s = \pm 1$.]

6 Transport theorems

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_t} f\,\mathrm{d}V = \int_{V_t}\left(\frac{\mathrm{D}f}{\mathrm{D}t} + f\nabla\cdot\boldsymbol{u}\right)\mathrm{d}V \quad \text{[for a material volume } V_t\text{]} \tag{F22}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{S_t} \boldsymbol{b}\cdot\,\mathrm{d}\boldsymbol{S} = \int_{S_t}\left(\frac{\partial\boldsymbol{b}}{\partial t} - \nabla\times(\boldsymbol{u}\times\boldsymbol{b}) + \boldsymbol{u}\nabla\cdot\boldsymbol{b}\right)\cdot\,\mathrm{d}\boldsymbol{S} \quad \text{[for a material surface } S_t\text{]} \tag{F23}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\gamma_t} \boldsymbol{a}\cdot\mathrm{d}\boldsymbol{l} = \int_{\gamma_t}\left(\frac{\partial\boldsymbol{a}}{\partial t} - \boldsymbol{u}\times\nabla\times\boldsymbol{a} + \nabla(\boldsymbol{u}\cdot\boldsymbol{a})\right)\cdot\mathrm{d}\boldsymbol{l} \quad \text{[for a material curve } \gamma_t\text{]} \tag{F24}$$