Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH44120-WE01

Title:
Geophysical and Astrophysical Fluids V

Time:	3 hours	
Additional Material provided:	Formula sheet	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

SECTION A

1. Consider an inertial frame defined by the orthogonal basis $\boldsymbol{e}_i = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ and a rotating frame with the orthogonal basis $\hat{\boldsymbol{e}}_i = \{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3\}$. The axis of rotation for the rotating frame is aligned with $\hat{\boldsymbol{e}}_2 = \frac{1}{2}\boldsymbol{e}_2 - \frac{\sqrt{3}}{2}\boldsymbol{e}_3$ and rotates with a constant angular velocity of $\Omega = 1$.

(a) Find an expression for $\left(\dfrac{d}{dt}(\hat{\boldsymbol{e}}_1 - 3\hat{\boldsymbol{e}}_2 + 4\hat{\boldsymbol{e}}_3)\right)_{\text{I}}$ (where subscript "I" denotes the inertial frame). Express your final answer in terms of $\hat{\boldsymbol{e}}_i$. Recall the relation between the time derivatives in the rotating and inertial frame:

$$\left(\frac{d\mathbf{X}}{dt}\right)_{\text{I}} = \left(\frac{d\mathbf{X}}{dt}\right)_{\text{R}} + \boldsymbol{\Omega} \times \mathbf{X}.$$ **[4]**

(b) If a particle's velocity vector is defined as $\boldsymbol{V} = 10\,t\,\boldsymbol{e}_2 + \exp(-t^2)\cos(t)\,\boldsymbol{e}_3$ in the inertial frame, find its acceleration in the rotating frame. Express your final answer in terms of $\boldsymbol{e}_i$. **[6]**

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2. Consider a fluid that is in hydrostatic and geostrophic balance (in the northern hemisphere), i.e.

$$-f_0 v = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}, \quad f_0 u = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}, \quad b = \frac{1}{\rho_0}\frac{\partial p}{\partial z},$$

where $f_0 > 0$ and ρ_0 are the Coriolis parameter and the average density respectively, and both are assumed constant.

(a) If the horizontal pressure gradient is in the south-east direction, find the direction of the horizontal velocity and justify your answer. **[3]**

(b) Find a relationship between buoyancy b and vertical vorticity $\zeta = \partial v/\partial x - \partial u/\partial y$. **[3]**

(c) If the buoyancy field for a flow is given as

$$b = -f_0\cos(x) + f_0\cos(y),$$

find its velocity field if $u = U_0$ and $v = V_0$ at $z = 0$. **[4]**

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3. Consider the magnetic field $\boldsymbol{B} = (\cos(\lambda z), \sin(\lambda z), 0)$, $\lambda \in \mathbb{R}$.

(a) For what value(s) of λ is $\boldsymbol{B}$ a force-free field? **[3]**

(b) For what value(s) of λ is $\boldsymbol{B}$ a potential field? **[2]**

(c) What type of pressure field is required to maintain a magnetostatic equilibrium? **[2]**

(d) Describe, using a sketch if helpful, the magnetic field lines for $\lambda \neq 0$. **[3]**

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4. The marginal Rayleigh number, Ra, for the onset of magnetoconvection (of a mode with vertical wavenumber $n = 1$) in an imposed background vertical magnetic field in a layer of fluid is

$$Ra = \frac{(\pi^2 + k_h^2)^3 + Q\pi^2(\pi^2 + k_h^2)}{k_h^2},$$

where k_h is the horizontal wavenumber and Q is a non-dimensional number characterising the strength of the magnetic field relative to viscous forces.

(a) Find the critical Rayleigh number in the absence of magnetic field, that is, for $Q = 0$. **[4]**

(b) For $Q \neq 0$, show that the critical horizontal wavenumber, k_{h_c}, satisfies

$$Q\pi^4 = 3(\pi^2 + k_{h_c}^2)^2 k_{h_c}^2 - (\pi^2 + k_{h_c}^2)^3.$$

Hence find the dependence of k_{h_c} and the critical Rayleigh number on Q in the asymptotic limit of $Q \to \infty$. **[4]**

(c) Briefly describe the physical effect of increasing Q on the onset of convection and the patterns you would expect to observe at onset. **[2]**

SECTION B

5. The two-dimensional non-hydrostatic Boussinesq equations (on $x - z$ plane) are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}, \tag{5.1a}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + b, \tag{5.1b}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{5.1c}$$

$$\frac{\partial b}{\partial t} + u\frac{\partial b}{\partial x} + w\frac{\partial b}{\partial z} + N^2 w = 0 \tag{5.1d}$$

where N is constant and ρ_0 is the average density.

(a) Consider a flow that is described by (5.1) and consists of a background horizontal velocity and small perturbations:

$$u = U_0 + u', \quad w = w', \quad p = p', \quad b = b', \tag{5.2}$$

where U_0 is the constant background velocity and u', w', p' and b' are perturbation variables. Derive a linearised system of equations that describes the time evolution of u', w', p' and b' assuming these variables are small and their products can be neglected. **[2]**

(b) Find the dispersion relation of the wave solution in the form of

$$u' = \tilde{u}e^{i(k_x x + k_z z - \omega t)}, \quad w' = \tilde{w}e^{i(k_x x + k_z z - \omega t)},$$
$$p' = \tilde{p}e^{i(k_x x + k_z z - \omega t)}, \quad b' = \tilde{b}e^{i(k_x x + k_z z - \omega t)},$$

for the linearised equations of part (a). **[5]**

(c) In the case of hydrostatic Boussinesq equation (i.e. when the left hand side of (5.1b) is neglected), the dispersion relation changes to

$$\omega = k_x U_0 + \omega_i, \quad \text{where} \quad \omega_i^2 = \frac{k_x^2 N^2}{k_z^2}. \tag{5.3}$$

(you don't need to show this). If the wave is steady (i.e. $\omega = 0$), find an expression for $|k_z|$ in terms of $|k_x|$, N and U_0 (you may assume $k_x \neq 0$). **[2]**

(d) For the hydrostatic Boussinesq and steady wave ($\omega = 0$), find a relationship between the sign of k_x and the sign of k_z, if: i) the wave energy is transferred upward (i.e. the positive direction in z), and ii) if the wave energy is transferred downward. Assume $U_0 > 0$ and use the dispersion relation (5.3) given in part (c). **[6]**

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6. The two-dimensional incompressible flow (on $x - y$ plane) in a rotating frame is described by

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - (f_0 + \beta y)v = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}, \qquad (6.1a)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + (f_0 + \beta y)u = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}, \qquad (6.1b)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \qquad (6.1c)$$

(a) Show that the evolution of vorticity $\zeta = \partial v/\partial x - \partial u/\partial y$ follows the equation:

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + \beta v = 0. \qquad (6.2)$$

Demonstrate all the steps in your derivation clearly. **[5]**

(b) Non-dimensionalise the equation (6.2) by assuming the length scale to be $\mathcal{L}$, the velocity scale to be $\mathcal{U}$ and the time scale to be $\mathcal{L}/\mathcal{U}$. Note that you need to find the scale of vorticity using its definition. Then find $\mathcal{L}$ such that all the terms become equally important, i.e. have the same order of magnitude. **[5]**

(c) For a turbulent flow that is described with (6.1), assume that the energy spectrum E depends only on β and wavenumber k. Hence, it can be written as

$$E = C\beta^m k^n \qquad (6.3)$$

where C is a dimensionless constant. Find the exponents m and n. Hint: the dimension of E is L^3/T^2 and the dimension of β is $1/(LT)$, where L and T are units of length and time, respectively. **[5]**

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7. (a) Derive the mean field induction equation for constant diffusivity η:

$$\frac{\partial \langle \boldsymbol{B} \rangle}{\partial t} = \nabla \times (\langle \boldsymbol{u} \rangle \times \langle \boldsymbol{B} \rangle) + \nabla \times \boldsymbol{\epsilon} + \eta \nabla^2 \langle \boldsymbol{B} \rangle,$$

where $\boldsymbol{\epsilon} = \langle \boldsymbol{u}' \times \boldsymbol{B}' \rangle$, for mean fields $\langle \boldsymbol{B} \rangle$ and $\langle \boldsymbol{u} \rangle$ and fluctuating fields $\boldsymbol{B}'$ and $\boldsymbol{u}'$ defined such that $\langle \boldsymbol{B}' \rangle = \langle \boldsymbol{u}' \rangle = 0$. **[5]**

(b) Now consider a mean magnetic field $\langle \boldsymbol{B} \rangle(x, y, t)$ and an incompressible mean velocity field $\langle \boldsymbol{u} \rangle(x, y, t)$ that are invariant in z. By writing $\langle \boldsymbol{u} \rangle = \boldsymbol{u}_h + w\boldsymbol{e}_z$ and $\langle \boldsymbol{B} \rangle = \boldsymbol{B}_h + B_z\boldsymbol{e}_z$, where $\boldsymbol{u}_h \cdot \boldsymbol{e}_z = 0$ and $\boldsymbol{B}_h = \nabla \times (A\boldsymbol{e}_z)$, show that the mean field induction equation can be written as two equations for A and B_z:

$$\frac{\partial A}{\partial t} = -(\boldsymbol{u}_h \cdot \nabla)A + \eta \nabla^2 A + \boldsymbol{\epsilon} \cdot \boldsymbol{e}_z,$$

$$\frac{\partial B_z}{\partial t} = -(\boldsymbol{u}_h \cdot \nabla)B_z + (\boldsymbol{B}_h \cdot \nabla)w + \eta \nabla^2 B_z + (\nabla \times \boldsymbol{\epsilon}) \cdot \boldsymbol{e}_z.$$

[8]

(c) Explain briefly how in principle Cowling's Theorem can be circumvented in mean field theory. **[2]**

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8. (a) By writing $\boldsymbol{B} = \nabla \times (\nabla \times (P\boldsymbol{x})) + \alpha \nabla \times (P\boldsymbol{x})$ in spherical coordinates, where α is a constant and $\boldsymbol{x} = r\boldsymbol{e}_r$ is the position vector, show that if P satisfies $\nabla^2 P + \alpha^2 P = 0$, then $\boldsymbol{B}$ is a linear force-free field.

You may use without proof that $\nabla^2(P\boldsymbol{x}) = \boldsymbol{x}\nabla^2 P + 2\nabla P$. **[4]**

(b) Consider an axisymmetric system in which $P(r, \theta) = \sum_{l=0}^{\infty} R_l(r) P_l(\cos\theta)$, where $P_l(\cos\theta)$ are the Legendre polynomials. Find the general form of $R_l(r)$ such that $\nabla^2 P + \alpha^2 P = 0$.

Hint: The spherical Bessel equation $x^2 u'' + 2xu' + [x^2 - l(l+1)]u = 0$ has solutions $u(x) = j_l(x)$ and $y_l(x)$ where $j_l(x)$ and $y_l(x) \to 0$ as $x \to \infty$; $j_l(0) = \delta_{l0}$ and $|y_l(x)| \to \infty$ as $x \to 0$. **[7]**

(c) For a magnetic field confined to $r < R$ with the boundary condition $B_r(R, \theta) = 0$, what values can α take? **[4]**
