Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2026	MATH4431-WE01

Title:
Advanced Probability IV

Time:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. The indicative marks shown in brackets for the main parts of each question are given as a guide to the weighting the markers expect to apply. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2026

SECTION A

1. A student asks their professor for n bespoke reference letters and provides n correctly addressed envelopes. The professor agrees, writes the letters, but–being absent-minded–places them into the envelopes uniformly at random, one per envelope. Let S_n be the number of letters that end up in their correct envelopes. By using inclusion-exclusion or otherwise, find $\mathsf{P}(S_n > 0)$ and show that $\mathsf{P}(S_n = 0) \to e^{-1}$ as $n \to \infty$. Deduce that, as $n \to \infty$, the distribution of S_n converges to $\mathsf{Poi}(1)$, the Poisson distribution with parameter 1.

Hint: What is the probability that letters 1, 2, ..., k are placed into their correct envelopes? **[10]**

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2. Given a sample $\{X_k\}_{k=1}^{3}$ from $\mathcal{U}(0,1)$, the uniform distribution on $(0,1)$, let $X_{(1)}$, $X_{(2)}$, and $X_{(3)}$ be the corresponding order variables.

(a) Carefully derive the probability density function $f_{X_{(2)}}(y)$. **[5]**

(b) Find the conditional density $f_{X_{(1)},X_{(3)}\,|\,X_{(2)}}(x, z\,|\,y)$ of $X_{(1)}$ and $X_{(3)}$ given that $X_{(2)} = y$. Explain your findings. **[5]**

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3. (a) For each of the following graph properties, determine if they are increasing, decreasing or non-monotone, and justify your answer.

(i) $\mathcal{P}_1 = \{G : G \text{ is bipartite}\}$

(ii) $\mathcal{P}_2 = \{G : G \text{ contains an induced subgraph } H\}$ for a fixed H

(iii) $\mathcal{P}_3 = \{G : \text{the largest component of } G \text{ is a tree}\}$

(iv) $\mathcal{P}_4 = \{G : G \text{ is non-planar}\}$

Note: a graph $G = (V, E)$ is *bipartite* if V can be split into two disjoint sets A, B (i.e., $V = A \sqcup B$) such that every edge $e \in E$ connects a vertex in A to one in B. **[3]**

(b) Let K_5 denote the complete graph on 5 vertices. Let $G_{n,p}$ be the binomial random graph and $G_{n,m}$ the uniform random graph on the vertex set $V = \{1, \dots, n\}$.

Let

- X_p be the number of 5-vertex subsets $S \subset V$ such that the subgraph induced by S is a K_5 in $G_{n,p}$,
- X_m be the analogous number for $G_{n,m}$.

(i) Compute the expectations $\mathsf{E}[X_p]$ and $\mathsf{E}[X_m]$. **[3]**

(ii) Find a relation between $p = p(n)$ and $m = m(n)$ that ensures

$$\lim_{n\to\infty} \frac{\mathsf{E}[X_p]}{\mathsf{E}[X_m]} = 1.$$ **[4]**

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4. We write $X \preceq Y$ for *stochastic domination*, meaning

$$\mathsf{P}(X > t) \leq \mathsf{P}(Y > t) \quad \text{for all } t \in \mathbb{R}.$$

Let $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ be one-dimensional Gaussian random variables with means μ_X, $\mu_Y \in \mathbb{R}$ and variances $\sigma_X^2, \sigma_Y^2 > 0$.

(a) Give necessary and sufficient conditions on (μ_X, σ_X) and (μ_Y, σ_Y) for the stochastic domination

$$X \preceq Y.$$

[5]

(b) Now assume $\mu_X = \mu_Y = 0$. Prove that

$$|X| \preceq |Y| \quad \Longleftrightarrow \quad \sigma_X \leq \sigma_Y.$$

[5]

SECTION B

5. Let $(X_n)_{n\geq 1}$ be independent random variables with common distribution satisfying

$$\lim_{x\to\infty} x e^{x^2/2} \mathsf{P}(X_1 > x) = A,$$

where $A > 0$ is a fixed constant.

(a) Find real $c > 0$ such that $\mathsf{P}\Big(\limsup_{n\to\infty} \frac{X_n}{\sqrt{\log n}} = c\Big) = 1.$ **[6]**

(b) Define $M_n = \max_{1\leq k\leq n} X_k$, the running record value at time n. Show that with the same constant c as above,

$$\mathsf{P}\Big(\limsup_{n\to\infty} \frac{M_n}{\sqrt{\log n}} = c\Big) = 1\,.$$ **[5]**

(c) Express the probability $\mathsf{P}(M_n \leq x)$ using the distribution of individual variables X_k, and use your result to find a constant c such that

$$\mathsf{P}\Big(\lim_{n\to\infty} \frac{M_n}{\sqrt{\log n}} = c\Big) = 1\,.$$ **[4]**

In your answer you should clearly state every result you use.
Hint: You may use without proof the following fact:
If $(x_n)_{n\geq 1}$ are real numbers, $m_n \equiv \max_{1\leq k\leq n} x_k$, and a monotone sequence $(b_n)_{n\geq 1}$ increases to infinity as $n \to \infty$, then the sets $\{n \in \mathbb{N} : x_n \geq b_n\}$ and $\{n \in \mathbb{N} : m_n \geq b_n\}$ are both finite or both infinite.

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6. Consider $S_0, S_1, \ldots$ an infinite simple random walk with parameter $p \neq 1/2$ starting at the origin, $S_0 = 0$. Write N for the total number of visits to the origin and write L for the time of the last visit to the origin,

$$N = \sum_{n\geq 0} \mathbb{1}_{\{S_n=0\}}, \qquad L = \sup\{n \geq 0 : S_n = 0\}.$$

(a) Find the distribution of N and deduce that $\mathsf{E}[e^{aN}] < \infty$ for some $a > 0$. **[6]**
Hint: You may use without proof the following fact:
for t small enough, $\Psi(t) := \sum_{n\geq 0} t^n \mathsf{P}(S_n = 0) = \big(1 - 4p(1-p)t^2\big)^{-1/2}$.

(b) Find the generating function of L and deduce that $\mathsf{E}[e^{aL}] < \infty$ for some $a > 0$. **[5]**

(c) By using the exponential Markov inequality or otherwise, show that N and L are finite random variables, $\mathsf{P}(N < \infty) = 1$ and $\mathsf{P}(L < \infty) = 1$. **[4]**

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7. Let $G_{n,p}$ be the binomial random graph on vertex set $V = \{1, \ldots, n\}$. An *isolated cherry* is a set of three vertices $\{u, v, w\}$ such that

- the induced subgraph $G[\{u, v, w\}]$ is a path of length 2 with centre v (i.e. edges uv and vw are present but uw is absent), and
- there are no edges between $\{u, v, w\}$ and $V \setminus \{u, v, w\}$ (so $\{u, v, w\}$ forms a connected component of size 3).

Let X be the number of isolated cherries in $G_{n,p}$.

(a) Compute the expectation $\mathsf{E}[X]$ explicitly in terms of n and p. **[4]**

(b) Compute the variance $\mathsf{Var}[X]$ explicitly in terms of n and p. **[5]**

(c) Let $p = p(n)$ satisfy $\dfrac{np}{\log n} \to c$ as $n \to \infty$, for some constant $c > 0$. Show that

$$\mathsf{P}(X \geq 1) \to 0 \quad \text{if } c > \tfrac{1}{3}, \qquad \text{and} \qquad \mathsf{P}(X \geq 1) \to 1 \quad \text{if } 0 < c \leq \tfrac{1}{3}.$$

(Hint: You may use the first and second moment methods.) **[6]**

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8. For bond percolation on the hexagonal (or honeycomb) lattice, show that its critical value p_{cr} is non-trivial: $0 < p_{\mathsf{cr}} < 1$. **[15]**

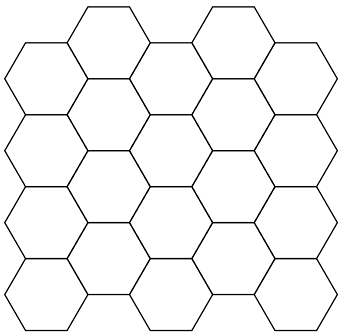

Figure 1: A small patch of the hexagonal (honeycomb) lattice.
