Hints for exercises 8.2 and 8.3 (week 11)

8.2 First, note that under the risk-neutral probabilities on $M(t)$, we have

$$\frac{S(t)}{S(0)} = \frac{e^{-ft}M(t)}{M(0)} = e^{-ft}e^{W}$$

where $W \sim N((r - \frac{1}{2}\sigma^2)t, \sigma^2 t)$. Thus, under the risk-neutral probabilities, we have $S(t) = S(0)e^{W-ft}$. Now, determine the distribution of $W - ft$ and from this you can determine the distribution of $S(t)$.

8.3 We suppose that the dividend payments at time t_1 and t_2 are used to purchase additional shares. Denote by $M(y)$ the market value of our portfolio at time y and $S(y)$ the price of a share at time y. We consider 3 diffent cases:

$y < t_{d_1}$ Determining $S(y)$ and $M(y)$ should not be a problem.

$t_1 \leq y < t_2$ First we have to determine the price of a share immediately after the first dividend is paid (remember that the price of a share decreases with exactly the dividend paid at a dividend payment time). Then, by determining how many additional shares you can buy from the dividend payment, you know how many shares you have at time t_1 and consequently, you can determine the value of $M(y)$ for $t_1 < y < t_2$.

$y > t_2$ Determine the price of a share immediately after the second dividend is paid and determine how many additonal shares you can buy from the dividend payment. Then you know how many shares you have and you can determine the value of $M(y)$ for $y > t_2$.

If you have done all this correctly, you should get

$$M(y) = \begin{cases} S(y) & \text{if } y < t_1 \\ \frac{1}{1-f}S(y) & \text{if } t_1 < y < t_2 \\ (\frac{1}{1-f})^2 S(y) & \text{if } y > t_2. \end{cases}$$

Now use the model where $M(y)$ is a Geometric Brownian Motion with volatility parameter σ. The risk-neutral probabilities for this process are that of a GBM with volatility parameter σ and drift parameter $r - \frac{1}{2}\sigma^2$. For each of the 3 cases described above, find an expression that relates $S(y)/S(0)$ to $M(y)/M(0)$. Then, using the fact that $M(y)/M(0)$ can be written as e^W with $W \sim N((r - \frac{1}{2}\sigma^2)y, \sigma^2 y)$, write $S(y)$ as function of W and you can determine the no-arbitrage cost, under the risk-neutral probabilities, of a (K, y) call option. Write $S(0) = s$, then you should get: No-arbitrage cost of a (K, y) option equals

$$e^{-ry}E[(S(y) - K)^+] = \begin{cases} C(s, y, K, \sigma, r) & \text{if } y < t_1 \\ C(s(1-f), y, K, \sigma, r) & \text{if } t_1 \leq y < t_2 \\ C(s(1-f)^2, y, K, \sigma, r) & \text{if } y \geq t_2. \end{cases}$$