Hints for exercises 8.7, 8.8 and 8.12 (week 13)

8.7 General procedure: Find a portfolio such that the payoff of this portfolio is the same as the payoff of the investment given in the exercise. Then, use the Law of One Price to determine the value of the investment c for there not to be an arbitrage. As the expression for K is in terms of the Black-Scholes formula, the portfolio should be a combination of European call and put options together with some cash.

The following portfolio together with the put-call option parity option formula yields the required result.

Buy a European $((1+\beta)s, 1)$ put option
Sell a European $(K, 1)$ put option
Cash amount of Ke^{-r}.

Note: Finding such a portfolio is not straightforward but there are many examples in the book where the Law of One Price is used.

8.8 Use the following two things:

- When the price of a security follows a Brownian motion with drift parameter $(r-\sigma^2/2)$ and volatility parameter σ, the unique no-arbitrage cost of a call option $C(s,t,K,\sigma,r)$ to purchase the security at time t for the price K is $e^{-rt}E[(S(t)-K)^+] = e^{-rt}E[(se^W - K)^+]$ where $W \sim N((r-\sigma^2/2)t, \sigma^2 t)$ and $s = S(0)$.
- Let Y be a normal random variable with mean $(r-f-\sigma^2/2)t$ and variance $\sigma^2 t$ then Y can be written as $Y = X - ft$ with $X \sim N((r-\sigma^2/2)t, \sigma^2/2)$.

Then $e^{-ft}C(s,t,K,\sigma,r-f) = e^{-ft}e^{-(r-f)t}E[(S(t)-K)^+] = e^{-rt}E[(se^W - K)^+)$ where $W \sim N((r-f-\sigma^2/2)t, \sigma^2 t)$, the result follows by writing $W = Z - ft$ with $Z \sim N((r-\sigma^2/2)t, \sigma^2/2)$.

8.12 The properties in Remark 1 should be proved if you wish to use them.

c & d) Part d) is true, $V_k(i) \geq V_k(i+1)$ for fixed k.
This clearly holds at stage n so assume it for some $k+1 \leq n$. Write $R_k(i) = \max(0, K - u^i d^{k-i} s)$. Then as $R_k(i) > R_k(i+1)$ and V_{k+1} is non-increasing in i we have

$$\begin{aligned} V_k(i) &= \max\Big(R_k(i), \beta p V_{k+1}(i+1) + \beta(1-p)V_{k+1}(i)\Big) \\ &\geq \max\Big(R_k(i+1), \beta p V_{k+1}(i+2) + \beta(1-p)V_{k+1}(i+1)\Big) = V_k(i+1) \end{aligned}$$

so V_k is also non-increasing. Now work backwards down to $k=1$.

a & b) Part a) is true, $V_k(i) \leq V_{k+1}(i)$ for fixed i.
This time we use $R_{k+1}(i) > R_k(i)$. Start with $V_{n-1}(i) = \max\big(R_{n-1}(i), \beta p R_n(i+1) + \beta(1-p)R_n(i)\big) < R_n(i) \equiv V_n(i)$ (remember that $R_n(i+1) < R_n(i)$). Now assume $V_j(i) \leq V_{j+1}(i)$ for fixed i and $j \geq k+1$ for some k. Then

$$\begin{aligned} V_k(i) &= \max\Big(R_k(i), \beta p V_{k+1}(i+1) + \beta(1-p)V_{k+1}(i)\Big) \\ &\leq \max\Big(R_{k+1}(i), \beta p V_{k+2}(i+1) + \beta(1-p)V_{k+2}(i)\Big) = V_{k+1}(i) \end{aligned}$$

and again work backwards down to $k = i$ for each i.