Hints for exercises 8.13, 8.14, 8.15 (week 14)

8.13 Using the put-call option parity formula, the risk-neutral price of a European put option satisfies

$$P(s,t,K,\sigma,r) = C(s,t,K,\sigma,r) + Ke^{-rt} - s$$

where

$$s = 9,\ t = \tfrac{1}{4},\ K = 10,\ \sigma = 0.3,\ r = 0.06$$

$$C(s,t,K,\sigma,r) = s\Phi(\omega) - Ke^{-rt}\Phi(\omega - \sigma\sqrt{t})$$

$$\omega = \frac{(r+\frac{1}{2}\sigma^2)t - \log(\frac{K}{S(0)})}{\sigma\sqrt{t}} = -0.5274034,\ \Phi(\omega) = \Phi(-0.5274034) = 0.298957$$

$$\omega - \sigma\sqrt{t} = -0.677403,\ \Phi(\omega - \sigma\sqrt{t}) = \Phi(-0.677403) = 0.2490751$$

Hence, $C(s,t,K,\sigma,r) = 0.2369444$ and $P(s,t,K,\sigma,r) = 1.0880638$.

8.14 Note: I have used $n = 5$ which is too small for an accurate approximation but it shows how to calculate the risk-neutral price of an American put option. For larger n, you could use the spreadsheet linked via the course page.

With the values of the parameters given in the exercise, we have

$$\begin{aligned} u &= e^{\sigma\sqrt{t/n}} = 1.0693832, \quad d = e^{-\sigma\sqrt{t/n}} = 0.9351184 \\ p &= \tfrac{1+rt/n-d}{u-d} = 0.50558, \quad 1-p = 0.49442, \quad \beta = e^{-rt/n} = 0.9970045 \end{aligned}$$

We have to calculate $V_0(0)$ which we will do by dynamic programming. Below you find the values of $V_k(i)$ for $k = 1,\ldots,5$ and $i = 0,\ldots,k$:

$$\begin{aligned}
&V_5(0) = 2.849556,\ V_5(1) = 1.8228914,\ V_5(2) = 0.6488177,\ V_5(i) = 0, i \ge 3 \\
&V_4(0) = 2.3534345,\ V_4(1) = 1.2555366,\ V_4(2) = 0.3198275,\ V_4(3) = V_4(4) = 0 \\
&V_3(0) = 1.8228906,\ V_3(1) = 0.7801169,\ V_3(2) = 0.1576554,\ V_3(3) = 0 \\
&V_2(0) = 1.2918038,\ V_2(1) = 0.4640166,\ V_2(2) = 0.0777144 \\
&V_1(0) = 0.8706713,\ V_1(1) = 0.2679038 \\
&V_0(0) = 0.5642263.
\end{aligned}$$

8.15 An asset-or-nothing call option pays F if the stock price is larger than K, otherwise nothing. Clearly, an American asset-or-nothing call option is exercised as soon as the share price is larger than K – interest is lost if we wait any longer. Using an n-step approximation let $V_k(i)$ be the time $t_k = kt/n$ expected return from this option, given that it has not been exercised, that the stock price is $S(t_k) = u^i d^{k-i} s$ and that an optimal policy will be followed from time t_k onwards. Write I_{ki} for the indicator function of the event $u^i d^{k-i} s > K$. Clearly $V_n(i) = F \cdot I_{ni}$ for each i. The one-step optimality equation can be written

$$V_k(i) = \max\big\{\beta[pV_{k+1}(i+1) + (1-p)V_{k+1}(i)], I_{ki}F\big\}$$

and because we exercise the option at the first opportunity it simplifies to

$$V_k(i) = \beta[pV_{k+1}(i+1) + (1-p)V_{k+1}(i)](1 - I_{ki}) + F \cdot I_{ki}, \quad i = 0,1,\ldots,k.$$

for each $k = n$, $n-1$, …, 0. Our estimate for the value of the option is $V_0(0)$.