Hints to exercises 9.1, 9.4, 9.6, 9.7 & 9.10 (week 15)

9.1 To have no arbitrage there must exist $p = (p_{50}, p_{175}, p_{200})$ such that both buying stocks and buying options are fair. This means we have to solve

$$E[G_0] = (50-c)p_{200} + (25-c)p_{175} - cP_{50} = 50p_{200} + 25p_{175} - c = 0 \quad (1)$$
$$E[G_s] = 100p_{200} + 75p_{175} - 50p_{50} = 0 \qquad (2)$$
$$p_{200} + p_{175} + p_{50} = 1 \quad (3) \qquad\qquad 0 \le p_{50}, p_{175}, p_{200} \le 1. \quad (4)$$

Let $p_{50} = x$. Solving p_{175} and p_{200} in terms of x from (2) and (3) yields

$$p_{175} = 4 - 6x, \qquad p_{200} = 5x - 3.$$

Hence, we have the solution $(p_{50}, p_{175}, p_{200}) = (x, 4-6x, 5x-3)$. Together with (4) this yields $\frac{3}{5} \le x \le \frac{2}{3}$.

From (1) we have $c = 25p_{175} + 50p_{200} = 100x - 50$. This together with $\frac{3}{5} \le x \le \frac{2}{3}$ yields $10 \le c \le \frac{50}{3}$.

9.4 Using the same notation as in the example, we get

$$E[U(X)] = \log(x) + p\log(1+r+\alpha-\alpha r) + (1-p)\log(1+r) + (1-p)\log(1-\alpha)$$

$$\frac{\partial E[U(X)]}{\partial \alpha} = \frac{2p-1-(r+\alpha-\alpha r)}{(1+r+\alpha-\alpha r)(1-\alpha)}.$$

As $0 \le \alpha, r \le 1$, we have $r + \alpha \ge \alpha r$. If $p \le \frac{1}{2}$, we have $2p - 1 \le 0$. Hence $\dfrac{\partial E[U(X)]}{\partial \alpha} < 0$ and max $E[U(X)]$ is obtained at $\alpha = 0$.

9.6 From the book max $E[U(W)]$ is equivalent to max $E[W] - \frac{1}{2}b\mathrm{Var}[W]$.

$$E[W] = 118 - 0.02y, \qquad \mathrm{Var}[W] = 0.1425y^2 - 16.5y + 625,$$
$$\max\ E[W] - \tfrac{1}{2}b\mathrm{Var}[W] = \max\ 116.4375 + 0.02125y - 3.5625 \cdot 10^{-4}y^2$$
$$\text{giving} \quad y = 29.824561, \qquad 100 - y = 70.175439$$
$$E[W] = 117.40351 \quad \text{and } \mathrm{Var}[W] = 259.64913$$
$$\max\ E[U(W)] = 1 - \exp\{-0.005 \cdot 117.40351 + 0.005^2 \cdot 259.64913/2\} = 0.4422.$$

(a) Invest 100 in security 1, i.e. $y = 100$. Then

$$E[W] = 116, \qquad \mathrm{Var}[W] = 400, \qquad E[U(W)] = 0.4372951.$$

(b) Invest 100 in security 2, i.e. $y = 0$. Then

$$E[W] = 118, \qquad \mathrm{Var}[W] = 625, \qquad E[U(W)] = 0.441325.$$

9.7 Prove that the optimal proportion of one's wealth w that should be invested in security i does not depend on w.

Proof: $W = w\sum_{i=1}^{n} \alpha_i X_i$ for any portfolio $w\alpha_1,\ w\alpha_2, \dots, w\alpha_n$. If $U(x) = \log(x)$ then

$$\begin{aligned} E[U(W)] &= E[\log(W)] = E[\log(w\sum_{i=1}^{n}\alpha_i X_i)] = E[\log(w) + \log(\sum_{i=1}^{n}\alpha_i X_i)] \\ &= \log(w) + E[\log(\sum_{i=1}^{n}\alpha_i X_i)] \end{aligned}$$

and so the optimal $\alpha_i,\ i = 1, \dots, n$ do not depend on w.

9.10 Find the portfolio that maximises $E[U(W)]$ by using the approximation, i.e. by maximising $U(E[W]) + U''(E[W]) \cdot \mathrm{Var}[W]/2$. We have

$$U(x) = 1 - e^{-0.005x}, \qquad U'(x) = 0.005e^{-0.005x}, \qquad U''(x) = -2.5 \cdot 10^{-5} e^{-0.005x}.$$

Suppose $w_1 = y$ and $w_2 = 100 - y$ then

$$\begin{aligned} &E[W] = 118 - 0.03y, \qquad \mathrm{Var}[W] = 0.1425y^2 - 16.5y + 625, \\ &U(E[W]) + \tfrac{1}{2}U''(E[W]) \cdot \mathrm{Var}[W] = \\ &\qquad 1 - (1.78125 \cdot 10^{-6} y^2 - 2.0625 \cdot 10^{-4} y + 1.0078125) \exp\{-0.005(118 - 0.03y)\} \end{aligned}$$

Maximising $U(E[W]) + U''(E[W]) \cdot \mathrm{Var}[W]/2$ is equivalent to minimising $f(y)$ where

$$f(y) = e^{-0.005(118-0.03y)}(1.78125 \cdot 10^{-6} y^2 - 2.0625 \cdot 10^{-4} y + 1.0078125)\ .$$

Solving $\partial f(y)/\partial y = 0$ and checking that $\partial^2 f(y)/\partial y^2 \geq 0$, we find that $y = 15.577604$ (with $100 - y = 84.422396$) is the optimal value of $f(y)$ so that

$$E[W] = 117.53267, \qquad \mathrm{Var}[W] = 402.54883, \qquad E[U(W)] = 0.441573.$$

The approximation gives $U(E[W]) + \frac{1}{2}U''(E[W]) \cdot \mathrm{Var}[W] = 0.44158$.