Hints to exercises 9.11, 9.12, 9.14 & 9.16 (week 16)

9.11 End-of-period wealth is $W = \sum_{i=1}^{n} w_i X_i$. Here, if the amount w_i is invested in security i then, after one period, that investment returns $w_i X_i$ and $w_1 + \cdots + w_n = w$ where w is the initial wealth. We assume W is a Normal random variable. The objective is

$$\max P(W > g) = \max P\left(\frac{W - E[W]}{\sqrt{\mathrm{Var}[W]}} > \frac{g - E[W]}{\sqrt{\mathrm{Var}[W]}}\right) = \max \Phi\left(-\frac{g - E[W]}{\sqrt{\mathrm{Var}[W]}}\right),$$

where $\Phi(\cdot)$ is the cumulative normal distribution function. That is, we have to maximise

$$-\frac{g - E[W]}{\sqrt{\mathrm{Var}[W]}} = \frac{E[W] - g}{\sqrt{\mathrm{Var}[W]}}.$$

9.12 Using the previous exercise, we have to maximise $f(y) = (E[W] - g)/\sqrt{\mathrm{Var}[W]}$. Now setting $w_1 = y$, $w_2 = 100 - y$ we have

$$\mathrm{Var}[W] = 0.1425y^2 - 16.5y + 625, \quad E[W] = 118 - 0.03y,$$

so we have to do some fairly lengthy differentiation. Do this carefully and you should find

$$f'(y) = 0 \Leftrightarrow -16.5675y + 954.75 + 0.1425gy - 8.25g = 0$$

An even longer calculation leads to

$$f''(y) \le 0 \Leftrightarrow 0.1425y^2 - 16.5y + 625 \ne 0$$

(a) $g = 110 : y^* = 52.94$ to invest in security 1 and $100 - y^* = 47.06$ to invest in security 2. Check $0.1425(y^*)^2 - 16.5y^* + 625 = 150.865 \ne 0$.

(b) $g = 115 : y^* = 33\frac{1}{3}$ to invest in security 1 and $100 - y^* = 66\frac{2}{3}$ to invest in security 2. Check $0.1425(y^*)^2 - 16.5y^* + 625 = 157.19 \ne 0$.

(c) $g = 120 : y^* = 66.197$ to invest in security 1 and $100 - y^* = 33.803$ to invest in security 2. Check $0.1425(y^*)^2 - 16.5y^* + 625 = 157.191 \ne 0$.

(d) $g = 125 : y^* = 61.45$ to invest in security 1 and $100 - y^* = 38.55$ to invest in security 2. Check $0.1425(y^*)^2 - 16.5y^* + 625 = 149.165 \ne 0$.

9.14 CAPM: $r_i = r_f + \beta_i(r_m - r_f)$. We have $\beta_i = 0.80$, $r_m = 0.07$ and $r_f = 0.05$ so that $r_i = 0.10 + 0.80(0.07 - 0.05) = 0.066$ In case $r_f = 0.10$, we have $r_i = 0.10 + 0.80(0.07 - 0.10$.

9.15 From the book $\beta_i = \mathrm{Cov}(R_i, R_m)/v_m^2$ where R_i is the one-period rate of return for security i and v_m^2 is the variance of the one-period rate of return of the market.

Suppose R is the one-period rate of return for the portfolio, i.e. $R = \sum_{i=1}^{n} \alpha_i R_i$, then β, the beta of the portfolio satisfies

$$v_m^2 \beta = \mathrm{Cov}(R, R_m) = \mathrm{Cov}\left(\sum_{i=1}^{n} \alpha_i R_i, R_m\right) = \sum_{i=1}^{n} \mathrm{Cov}(\alpha_i R_i, R_m) = \sum_{i=1}^{n} \alpha_i \mathrm{Cov}(R_i, R_m)$$

and hence $\beta = \sum_{i=1}^{n} \alpha_i \beta_i$.

9.16 CAPM: $R_i = r_f + \beta_i(R_m - r_f) + e_i = r_f(1 - \beta_i) + \beta_i R_m + e_i$ where e_i is a normal random variable with mean 0. Moreover, e_i and R_m are independent for $i = 1, \ldots, n$. So, taking

$$a_i = (1 - \beta_i)r_f, \quad b_i = \beta_i, \quad F = R_m$$

yields the single-factor model given in the exercise.