Hints to exercises 11.1, 11.2, 11.3, 11.4 (week 17)

11.1 At time y, the price of the security equals $S(y)$. Suppose you own an option to buy one share of the stock at a price Ke^{uy} at time y. The option expires after an additional time $t-y$.
If you exercise the option at $y<t$ you realize the amount $S(y)-Ke^{uy}$, which is worth $(S(y)-Ke^{uy})e^{(t-y)r}$ at time t. Now, instead of exercising the option early, suppose you sell the stock short and purchase the stock at time t, either by paying the market price at that time or by exercising your option and paying Ke^{ut}, whichever is less expensive. Hence, in this case, your payoff is worth at time t $S(y)e^{(t-y)r}-\min(Ke^{ut},S(t))$.

As $S(y)e^{(t-y)r}-\min(Ke^{ut},S(t)) \geq (S(y)-Ke^{uy})e^{(t-y)r}$ $(\star)$ you never exercise the call option earlier if $u \geq r$. To see why $(\star)$ is true, first suppose $\min(Ke^{ut},S(t))=Ke^{ut}$. As $uy+tr-yr \geq ut$ (i.e. $(t-y)(r-u)\geq 0$), we have that $Ke^{ut} \leq Ke^{uy+tr-yr}$ and consequently $S(y)e^{(t-y)r}-Ke^{ut} \geq (S(y)-Ke^{uy}e^{(t-y)r}$ as required.
Now if $\min(Ke^{ut},S(t))=S(t)$ then $S(t)<Ke^{ut}\leq Ke^{uy+tr-yr}$ and hence
$S(y)e^{(t-y)r}-S(t) > S(y)e^{(t-y)r}-Ke^{uy+tr-yr}=(S(y)-Ke^{uy})e^{(t-y)r}$ as required.

11.2 Section 11.5 explains how to simulate a sequence of end-of-day prices and from these the payoff is easily calculated. Section 11.6.1 discusses the use of antithetic variables to reduce variation and improve efficiency.

11.3 $V=(S_d(n)-K)^+$ so $E(V)=E((S_d(n)-K)^+)=e^{rn/N}C(S_d(0),n/N,K,\sigma,r)$
where $C=S_d(0)\Phi(\omega)-Ke^{-rn/N}\Phi(\omega-\sigma\sqrt{n/N}$
and $\omega=rn/N+\frac{1}{2}\sigma^2\frac{n}{N}-\log(K/S_d(0))/\sigma\sqrt{n/N}$.

11.4 (a) $W=Y+\sum_{i=1}^{n}c_i(X_i-\mu_i)$ so

$$\begin{aligned}\mathrm{Var}(W) &= \mathrm{Var}(Y+\sum_{i=1}^{n}c_i(X_i-\mu_i))\\ &= \mathrm{Var}(Y)+\mathrm{Var}(\sum_{i=1}^{n}c_i(X_i-\mu_i))+2\mathrm{Cov}(Y,\sum_{i=1}^{n}c_i(X_i-\mu_i))\\ &= \mathrm{Var}(Y)+\sum_{i=1}^{n}c_i^2\mathrm{Var}(X_i)+2\sum_{i=1}^{n}c_i\mathrm{Cov}(Y,X_i).\end{aligned}$$

(b) We must differentiate with respect to the c_i.

$$\begin{aligned}\frac{\partial \mathrm{Var}(W)}{\partial c_i} &= 2c_i\mathrm{Var}(X_i)+2\mathrm{Cov}(Y,X_i)=0, \qquad i=1,\dots,n\\ \Rightarrow \quad c_i &= -\mathrm{Cov}(Y,X_i)/\mathrm{Var}(X_i) \qquad i=1,\dots,n\\ \text{and} \quad \frac{\partial^2 \mathrm{Var}(W)}{\partial c_i^2} &= 2\mathrm{Var}(X_i)\geq 0, \qquad \frac{\partial^2 \mathrm{Var}(W)}{\partial c_i \partial c_j}=0.\end{aligned}$$

Matrix of second derivatives is diagonal with $2\mathrm{Var}(X_i)$ as diagonal elements and hence is positive definite so this stationary point is a local minimum.