Hints to exercises 13.2, 13.4, 13.5, 13.6 (week 19)

13.2 Expected present value payoff of this call option equals

$$E(e^{-rn/N}(e^{L(n)} - K)^+) = e^{-rn/N}(e^{m(n)+v(n)/2}\Phi(\sqrt{v(n)} - h) - K\Phi(-h))$$

where $h = (\log K - m(n))/\sqrt{v(n)}$,

$$m_s(n) = m_s(60) = 4 + 0.7^{60}\log s, \qquad v(n) = v(60) = 0.1/0.51 = 0.1961,$$

and $h_{|s=48} = h_{|s=50} = h_{|s=52} = -0.19868$, $m_{|s=48} = m_{|s=50} = m_{|s=52} = 4$ so that the expected present value payoff of this call option equals 15.2225.

13.4

$$s^* = \exp\left\{\frac{a + \sigma^2/2N}{1-b}\right\} = 64.5001.$$

NB: $\sigma^2/N = 0.1$, i.e. $\sigma^2 = 0.1 * 252 = 25.2$.

13.5 $L(n) = a+bL(n-1)+e(n)$ is equivalent to $S_d(n) = e^{a+e(n)}(S_d(n-1))^b$. If $S_d(n-1) = s$ then $E[S_d(n)] = e^{a+\sigma^2/2N}s^b$. Let $0 < b < 1$ and $s^* = \exp\{\frac{a+\sigma^2/2N}{1-b}\}$. If $s > s^*$ then $s > \exp\{\frac{a+\sigma^2/2N}{1-b}\}$ which implies that

(i) $s^{1-b} > \exp\{a + \sigma^2/2N\}$ or $s > \exp\{a + \sigma^2/2N\}s^b = E[S_d(n)]$

(ii) $s^b > \exp\{\frac{a+\sigma^2/2N)b}{1-b}\} = \exp\{\frac{a+\sigma^2/2N}{1-b} - (a + \sigma^2/2N)\}$ or
$s^* = \exp\{\frac{a+\sigma^2/2N}{1-b}\} < s^b\exp\{a + \sigma^2/2N\} = E[S_d(n)]$.

Consequently, $s^* < E[S_d(n)] < s$.

13.6 Suppose $S_d(n-1) = s^*$. Show that $E[S_d(n)] = s^*$.

$$S_d(n) = e^{a+e(n)}(s^*)^b, \qquad E[e^{a+e(n)}] = e^{a+\sigma^2/2N}$$

and

$$\begin{aligned} E[S_d(n)] &= e^{a+\sigma^2/2N}\exp\left\{\frac{(a+\sigma^2/2N)b}{1-b}\right\} \\ &= \exp\left\{\frac{(1-b)(a+\sigma^2/2N) + b(a+\sigma^2/2N)}{1-b}\right\} \\ &= \exp\left\{\frac{a+\sigma^2/2N}{1-b}\right\} = s^*. \end{aligned}$$