Homework 4

1. Suppose the investor invests αx $(0 \leq \alpha \leq 1)$ and puts $(1-\alpha)x$ in the bank. After one period, she has $(1-\alpha)(1+r)x$ in the bank and the investment will be worth either $2\alpha x$ with probability p or 0 with probability $1-p$. The expected value of the utility of her fortune W is

$$\begin{aligned} E[U(W)] &= p\left[1-e^{-b\{(1-\alpha)(1+r)x+2\alpha x\}}\right] + (1-p)\left[1-e^{-b(1-\alpha)(1+r)x}\right] \\ &= 1-e^{-b(1-\alpha)(1+r)x}\left[pe^{-2\alpha xb}+1-p\right]. \end{aligned}$$

The stationarity condition

$$\frac{\partial E[U(W)]}{\partial \alpha} = -e^{-b(1-\alpha)(1+r)x}\left[bx(1+r)(pe^{-2\alpha xb}+1-p) - 2xbpe^{-2\alpha xb}\right] = 0$$

has a unique solution

$$\hat{\alpha} = -\frac{1}{2xb}\log\left(\frac{(1-p)(1+r)}{p(1-r)}\right).$$

In addition we have

$$\frac{\partial^2 E[U(W)]}{\partial \alpha^2} = -e^{-b(1-\alpha)(1+r)x}\cdot(a+c) \quad (\star)$$

where

$$\begin{aligned} a &= b(1+r)x\left[bx(1+r)(pe^{-2\alpha xb}+1-p) - 2xbpe^{-2\alpha xb}\right] \\ c &= bx(1+r)(-2xbpe^{-2\alpha xb}) + 4x^2b^2pe^{-2\alpha xb} = 2x^2b^2p(1-r)e^{-2\alpha xb} \end{aligned}$$

For $r < 1$ it is clear that $c \geq 0$ for all $\alpha \in [0,1]$. It is also evident that
$$a = -e^{b(1-\alpha)(1+r)x}b(1+r)x\frac{\partial E[U(W)]}{\partial \alpha}.$$
Hence, at $\alpha = \hat{\alpha}$ we have $a = 0$. Consequently, it follows from $(\star)$ that the second derivative is always less than 0 at $\alpha = \hat{\alpha}$ and hence a is a maximum.
This solution is only workable if $0 \leq \hat{\alpha} \leq 1$ i.e. we must have

$$-2xb \leq \log\left(\frac{(1-p)(1+r)}{p(1-r)}\right) \leq 0 \quad \text{or} \quad \frac{1+r}{2} \leq p \leq \frac{1+r}{(1-r)e^{-2xb}+(1+r)}$$

Hence,

$$\alpha^* = \begin{cases} 0, & p < (1+r)/2, \\ \hat{\alpha}, & \dfrac{1+r}{2} \leq p \leq \dfrac{1+r}{(1-r)e^{-2xb}+(1+r)}, \\ 1, & p > \dfrac{1+r}{(1-r)e^{-2xb}+(1+r)}. \end{cases}$$

Note that the solution depends upon x but only via the product bx which is appropriate for this utility function.

2. Many apologies, I meant to put $b = 0.01$, not 0.001! We are told that the return from any portfolio is Normal so we want to maximise $E(W) - b\text{Var}(W)/2$ where $b = 0.001$. Let $w_1 = y$, $w_2 = 100 - y$ so that
$$E(W) = 100 + 0.05y + 0.08(100 - y) = 108 - 0.03y,$$
$$\text{Var}(W) = [y^2 + 4(100 - y)^2 - 2y(100 - y)]/100 = (7y^2 - 1000y + 40000)/100.$$
Hence
$$g(y) \equiv E(W) - b\text{Var}(W)/2 = 108 - 0.03y - (7y^2 - 1000y + 40000)/20000$$
and so
$$g'(y) = -0.03 - (14y - 1000)/20000 = 0 \quad \text{at any stationary point.}$$
The only stationary point is at $y = (1000 - 600)/14 \approx 28.5714$ and must be a maximum as $g''(y) = -7b < 0$.
Using the value $b = 0.001$ gives $y = (1000 - 6000)/14 < 0$ so the optimal portfolio consists entirely of the second security.
A similar calculation with $b = 0.01$, $v_1 = 0.05$ and $v_2 = 0.1$ reduces the variance above to $(7y^2 - 1000y + 40000)/400$. The linear term is unchanged and the optimal $y = -1000$ and again the optimal portfolio consists entirely of the second security. The result is similar (even more extreme) with $b = 0.001$.

3. Suppose that for some $n \geq 1$ we have $L(n) = \sum_{i=0}^{n-1} b^i e(n-i) + \frac{a(1-b^n)}{1-b} + b^n L(0)$. This is certainly true for $n = 1$ so we can assume this. Now consider $L(n+1)$. We find that
$$\begin{aligned} L(n+1) &= a + b\left(\sum_{i=0}^{n-1} b^i e(n-i) + \frac{a(1-b^n)}{1-b} + b^n L(0)\right) + e(n+1) \\ &= \sum_{i=0}^{n-1} b^{i+1} e(n-i) + e(n+1) + a\frac{(1-b) + b(1-b^n)}{1-b} + b^{n+1} L(0) \\ &= \sum_{j=0}^{n} b^j e(n+1-j) + a\frac{1-b^{n+1}}{1-b} + b^{n+1} L(0) \qquad (\text{put } j = i+1) \end{aligned}$$
which is the appropriate formula at $n + 1$. By induction the formula holds for all finite n.