Homework 3

1. (a) There are 3 possible prices at time 1: 40 with probability p_{40}, 50 with probability p_{50} and 55 with probability p_{55}.

$$\begin{aligned}
&E[\text{present value return from purchasing one share of the stock}]\\
&= p_{40}\left(\frac{40}{1+r}-50\right)+p_{50}\left(\frac{50}{1+r}-50\right)+p_{55}\left(\frac{55}{1+r}-50\right)\\
&= \frac{10}{1+r}(4p_{40}+5(1-p_{40}-p_{55})+5.5\,p_{55})-50(p_{40}+p_{50}+p_{55})\\
&= \frac{10}{1+r}(-p_{40}+0.5p_{55}+5)-50=\frac{100}{21}(-2p_{40}+p_{55}-1/2).
\end{aligned}$$

(b)
$$\begin{aligned}
&E[\text{present value return from purchasing one call option}]\\
&= p_{55}\left(\frac{3}{1+r}-c\right)-p_{40}c-p_{50}c=p_{55}\frac{3}{1+r}-c=p_{55}\frac{20}{7}-c.
\end{aligned}$$

Hence the no-arbitrage cost of the call option satisfies $c=20p_{55}/7$.

(c) The risk-neutral probabilities are on the line $-2p_{40}+p_{55}=1/2$ and we need to find the extreme values of p_{55} on this line within the triangle $p_{40}+p_{55}\le 1$ with $p_{40}\ge 0$, $p_{55}\ge 0$. This can be done with a diagram, linear programming methods or by observing that $p_{55}=2p_{40}+1/2$ so that (i) $p_{40}\ge 0\Rightarrow p_{55}\ge 1/2$; (ii) $1\ge p_{40}+p_{55}=3p_{40}+1/2\Leftrightarrow p_{40}\le 1/6\Leftrightarrow p_{55}\le 5/6$.
Hence, as $c=20p_{55}/7$, we have $10/7\le c\le 50/21$.

(d) Suppose $c=10/7$. Let x_1 be the number of shares you buy and x_2 the number of call options you buy.
At time 1 price of 55 we have

$$\text{return } =x_1\left(\frac{55}{1+r}-50\right)+x_2\left(\frac{3}{1+r}-\frac{10}{7}\right)=\frac{10}{21}(5x_1+3x_2).$$

At time 1 price of 50 we have

$$\text{return } =x_1\left(\frac{50}{1+r}-50\right)-x_2\cdot\frac{10}{7}=-\frac{10}{21}(5x_1+3x_2).$$

At time 1 price of 40 we have

$$\text{return } =x_1\left(\frac{40}{1+r}-50\right)-x_2\cdot\frac{10}{7}=-\frac{10}{21}(25x_1+3x_2).$$

Take $x_1=-3$ and $x_2=5$, that is, sell short three shares and buy 5 call options, then return is 0 if price equals 55 or 50 and return equals 200/7 if the time 1 price equals 40.

2. (a) The Black-Scholes formula

$$C=S_0\Phi(\omega)-Ke^{-rt}\Phi(\omega-\sigma\sqrt{t}),\quad\text{with}\quad \omega=\frac{rt+\frac{1}{2}\sigma^2t-\log(K/S_0)}{\sigma\sqrt{t}}$$

and $\Phi(\cdot)$ the standard normal distribution function, gives us the risk-neutral valuation of this European call option. Substituting the parameters given in the exercise, we have

$$\begin{aligned} \omega = 0.4626, &\quad \omega - \sigma\sqrt{t} = 0.2788, \\ \Phi(\omega) = 0.6782, &\quad \Phi(\omega - \sigma\sqrt{t}) = 0.6098, \end{aligned}$$

(using interpolation in the Normal tables!) which results in
$C = 46.75 \cdot 0.6782 - 45 \cdot e^{-0.06 \cdot 1/2} \cdot 0.6098 = £5.0759$.

(b) We know that $\log\left(\frac{S_t}{S_0}\right) \sim N((r - \frac{1}{2}\sigma^2)t, \sigma^2 t)$ so that $\log(S_t) \sim N(\log(S_0) + (r - \frac{1}{2}\sigma^2)t, \sigma^2 t)$. Hence, a 95% confidence interval for $\log(S_t)$ is given by

$$\left(\log(S_0) + (r - \sigma^2/2)t - u_{0.025} \cdot \sigma\sqrt{t}, \log(S_0) + (r - \sigma^2/2)t + u_{0.025} \cdot \sigma\sqrt{t}\right).$$

Consequently, a 95% confidence interval for S_t is given by (where $u_{0.025} = 1.96$)

$$\left(S_0 \exp\left\{(r - \sigma^2/2)t - u_{0.025} \cdot \sigma\sqrt{t}\right\},\ S_0 \exp\left\{(r - \sigma^2/2)t + u_{0.025} \cdot \sigma\sqrt{t}\right\}\right).$$

(c) As $\sigma \to 0$, the stock is virtually riskless and the price will grow at rate r to $S_0 e^{rt}$ at time t and the present value of the payoff from this call option is

$$e^{-rt} \max[S_0 e^{rt} - K, 0] = \max[S_0 - K e^{-rt}, 0].$$

This is consistent with the Black-Scholes formula given at the start of the solution. To see this, first consider the case that $S_0 > K e^{-rt}$, that is, $rt - \log(\frac{K}{S_0}) > 0$. As $\sigma \to 0$ then $\omega \to \infty$ and $\Phi(\omega) \to 1$, so that $C = S_0 - K e^{-rt}$ which is consistent with $\max[S_0 - K e^{-rt}, 0] = S_0 - K e^{-rt}$.
Now consider the case that $S_0 < K e^{-rt}$, that is, $rt - \log(\frac{K}{S_0}) < 0$. As $\sigma \to 0$ then $\omega \to -\infty$ and $\Phi(\omega) \to 0$, so that $C = 0$ which is consistent with $\max[S_0 - K e^{-rt}, 0] = 0$.

(d) Denote by C_F the risk-neutral cost of the European asset-or-nothing call option. Then

$$\begin{aligned} C_F &= e^{-rt} \cdot F \cdot Pr(S_t > K) = e^{-rt} \cdot F \cdot P\left(\log\left(\frac{S_t}{S_0}\right) > \log\left(\frac{K}{S_0}\right)\right) \\ &= e^{-rt} \cdot F \cdot P\left(\frac{\log S_t/S_0 - (r - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} > \frac{\log K/S_0 - (r - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right) \\ &= e^{-rt} \cdot F \cdot P\left(Z > \frac{\log 45/46.75 - (0.06 - \frac{1}{2}0.26^2) \cdot \frac{1}{2}}{0.26\sqrt{1/2}}\right) \\ &= e^{-rt} \cdot F \cdot P(Z > -0.2788) = e^{-rt} \cdot F \cdot P(Z < 0.2788) \\ &= e^{-0.06 \times 0.5} \cdot F \cdot 0.6098 = 0.5918 \cdot F. \end{aligned}$$

(obviously the values 0.2788 and $\Phi(0.2788) = 0.6098$ are the same as in part a and it would have OK to say this). Equating this to the risk-neutral valuation of the normal European call option $C = £5.0759$ yields

$$5.0759 = 0.5918 \cdot F \Longrightarrow F = £8.5771.$$