Homework 4

1. An investor with capital x can invest any amount between 0 and x. If y is invested the return is either $2y$ or 0 with respective probabilities p and $1-p$. The investment must be paid immediately whereas the payoff (if any) occurs after one period. Suppose further that whatever amount is not invested can be put in a bank to earn interest at a rate of r per period. Suppose an investor has an exponential utility function, $u(x) = 1 - e^{-bx}$ for amounts $x > 0$ where $b > 0$ is a known constant. How much should be invested by this person if they wish to maximise their utility?

2. You have 100 units to invest in two securities with rates of return R_i that have the following expected values and standard deviations:

$$r_1 = 0.05,\ v_1 = 0.1, \qquad r_2 = 0.08,\ v_2 = 0.2$$

   The correlation between the rates of return is $\rho = -0.5$ and your utility function is $U(x) = 1 - e^{-x/1000}$. Suppose you invest amounts w_i in security i and that your end-of-period return $W = w_1(1 + R_1) + w_2(1 + R_2)$ is normally distributed. Find the portfolio (w_1, w_2) that maximises your end-of-period utility.

   What should you do if $v_1 = 0.05$ and $v_2 = 0.1$ instead of the values above?

3. Let $S_d(n)$ denote a given stock's end-of-day price and let $L(n) = \log S_d(n)$. Suppose that $L(0)$ is knows and $L(n)$ follows an autoregressive model of order 1, that is

$$L(n) = a + bL(n-1) + e(n)\ , \qquad n \geq 1$$

   where a and b are constants and the $e(n)$ are a sequence of independent normal random variables with mean 0 and variance σ^2/N (where N is the number of trading days in a year). Show by induction (or otherwise) that

$$L(n) = \sum_{i=0}^{n-1} b^i e(n-i) + \frac{a(1-b^n)}{1-b} + b^n L(0)$$

   (this is formula (13.3) in Ross).