Cohomogeneity One Actions

Jürgen Berndt
Department of Mathematics, University of Hull, Hull HU6 7RX, England

We consider the problem:

For a given connected complete Riemannian manifold M determine the space $\mathfrak{M}$ of all isometric cohomogeneity one actions on M modulo orbit equivalence.

For the sphere S^n it follows from work by Hsiang and Lawson that $\mathfrak{M}$ is isomorphic to the set of all connected $(n+1)$-dimensional Riemannian symmetric spaces of non-compact type and rank two, the bijective correspondence is given via the isotropy representation of the symmetric space. In successive work by Takagi, Uchida, Iwata and Kollross the problem was solved for all connected, simply connected, irreducible Riemannian symmetric spaces of compact type. For any such space $\mathfrak{M}$ is a finite set. For the Euclidean space $\mathbb{R}^n$ it can be seen easily that $\mathfrak{M}$ is a set of n points, and it follows from work of E. Cartan that for the real hyperbolic space $\mathbb{R}H^n$ the moduli space $\mathfrak{M}$ consists of $n+1$ points.

We study the above problem for connected irreducible Riemannian symmetric spaces M of non-compact type. Every cohomogeneity one action on M either induces a foliation on M or has exactly one singular orbit. This induces a disjoint union $\mathfrak{M} = \mathfrak{M}_F \cup \mathfrak{M}_S$, where $\mathfrak{M}_F$ is the space of all homogeneous codimension one foliations on M modulo isometric congruence and $\mathfrak{M}_S$ is the space of all connected normal homogeneous submanifolds with codimension ≥ 2 in M modulo isometric congruence.

Denote by r the rank of M. To each $\ell \in \mathbb{R}P^{r-1}$ we associate a homogeneous codimension one foliation on M all of whose leaves are isometrically congruent to each other. If $r \geq 2$ some of these foliations are harmonic, i.e. all leaves are minimal submanifolds. To each $i \in \{1, \ldots, r\}$ we associate a homogeneous codimension one foliation on M which has exactly one minimal leaf. If $r \geq 3$ then some of these foliations form homogeneous isoparametric systems on M having the same principal curvatures with the same multiplicities, but which are not isometrically congruent to each other. The symmetry group $Aut(DD)$ of the Dynkin diagram of the restricted root system Σ of M acts on a set of simple roots in Σ, which is a set of r elements. This action extends canonically to an action on $\mathbb{R}P^{r-1}$. In this way we get an action of the finite group $Aut(DD)$ on $\mathbb{R}P^{r-1} \cup \{1, \ldots, r\}$. We prove that $\mathfrak{M}_F$ is isomorphic to the orbit space $(\mathbb{R}P^{r-1} \cup \{1, \ldots, r\})/Aut(DD)$. A corollary is that on each of the hyperbolic spaces $\mathbb{R}H^n$, $\mathbb{C}H^n$, $\mathbb{H}H^n$ and $\mathbb{O}H^2$ there exist exactly two homogeneous foliations of codimension one, and that $\mathfrak{M}$ is not a finite set if $r \geq 2$.

The work on the moduli space $\mathfrak{M}_S$ is still in progress and the structure of $\mathfrak{M}_S$ is not yet clear. For some Hermitian symmetric spaces and some quaternionic Kähler symmetric spaces we can construct explicitly families of normal homogeneous submanifolds of codimension ≥ 2 which can be parametrized by the Kähler resp. quaternionic Kähler angle of the normal bundle. But for some other symmetric spaces $\mathfrak{M}_S$ seems to be finite.