OSCILLATION OF EIGENFUNCTIONS OF FOURTH-ORDER TWO-POINT BOUNDARY VALUE PROBLEMS

Lyonell Boulton
Heriot-Watt University

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1- We begin by quoting:

[Ince, 1926] (referring to Sturm's oscillation theorem)
"*It is important from the point of view of physical applications, and not without theoretical interest, to determine the number of zeros which the solution has in the interval (a,b).*"

2- Consider the regular Sturm-Liouville problem:

$$(*) \qquad \begin{cases} -(py')' + (q - \underset{\substack{\uparrow \\ \texttt{spectral parameter}}}{E})y = 0 \\ \sin(\alpha)y(0) - p(0)\cos(\alpha)y'(0) = 0 \\ \sin(\beta)y(1) - p(1)\cos(\beta)y'(1) = 0 \end{cases}$$

p', q continuous (or L^∞), $p > 0$, q real-valued.

Sturm's Theorem: [1826] (*) *has an infinite set of eigenvalues all real, simple and accumulating only at* $E = +\infty$. *If we arrange the eigenvalues in increasing order* $E_0 < E_1 < \ldots$ *and* $\{y_n\}$ *are the corresponding eigenfunctions, then* y_n *has exactly* n *zeros in* $(0, 1)$.

3- The fourth-order regular Sturm-Liouville e.v.p.

Let the eigenvalue problem:

$$(EQ) \qquad (py'')'' + (q - E)y = 0,$$

where p'', q are continuous, $p > 0$ and q is real-valued.

Let the separated b.c. given in general form:

$$(BC)_{MN} \qquad M\overline{y}_0 = 0 = N\overline{y}_1,$$

where $M, N \in \mathbb{R}^{2\times 4}$, $\qquad \overline{y}_a := [(py'')', py'', y', y]^t|_{x=a}$.

$$\boxed{\text{Below we always assume that } M \text{ and } N \text{ are of maximal range and are given in reduced row form.}}$$

(EQ) $$(py'')'' + (q-E)y = 0$$

$(BC)_{MN}$ $$M\overline{y}_0 = 0 = N\overline{y}_1$$

$\boxed{\text{Q}}$ Can we recover oscillation results in the spirit of Sturm's Theorem when conditions on the coefficients of M and N ensure self-adjointness?

[Atkinson, 1964]: conjugate points...

But what about the counting of zeros?

4- A very simple example. The equation $y^{(4)} - Ey = 0$ with

$$M = N = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In this case $E_n = (n\pi)^4$ and $y_n(x) = \sin(n\pi x)$. Notice that if $\mathcal{L}y := -y''$ with Dirichlet b.c., then the above e.v.p. is the one associated to $\mathcal{L}^2$.

5- Background.

[S. Janczewsky, 1928]: Sturm's Theorem is completely recovered for a *special class* of M and N, and sufficiently large eigenvalues. This class includes

$$M = N = D := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

[W. Leighton, Z. Nehari, 1958], [K. Kreith 1970's]: interlacing and comparison results for (EQ).

$\boxed{\text{Q}}$ Universal estimates for general self-adjoint b.c.?

6- Weak formulation, self-adjointness.

$$a(u,v) := \int_0^1 p u'' v'' \,\mathrm{d}x, \qquad c(u,v) := \int_0^1 q u v \,\mathrm{d}x,$$

and $$b(u,v) := \left[(pu'')'v - pu''v'\right]\Big|_{x=0}^{x=1}.$$

Denote by $\tilde{M}, \tilde{N}$ the reduced boundary conditions. Let
$\tilde{\mathcal{D}} := \{y \in H^2(0,1) : \tilde{M}\overline{y}_0 = 0 = \tilde{N}\overline{y}_1\}$ and
$\mathcal{D} := \{y \in H^4(0,1) : M\overline{y}_0 = 0 = N\overline{y}_1\}$.

$$\text{(s)} \quad \boxed{\begin{vmatrix} r_{11} & r_{14} \\ r_{21} & r_{24} \end{vmatrix} = \begin{vmatrix} r_{12} & r_{13} \\ r_{22} & r_{23} \end{vmatrix} \quad \text{for } [r_{ij}] = M \text{ and } N.}$$

The form $a + c + b$ is symmetric, bounded below and closable in the domain $\mathcal{D} \subset L^2(0,1)$. Its closure has domain $\tilde{\mathcal{D}}$.

Notice that (s) is necessary and sufficient for the first statement to hold true.

7- Interlacing properties of the eigenvalues.

$(EQ)-(BC)_{MN}$ has an infinite set of eigenvalues, they are of multiplicity no greater than two and satisfying

$$E_n = \alpha_p n^4 + O(n^3) \quad \text{as} \quad n \to \infty, \quad \alpha_p := \left[\frac{2\pi}{\int_0^1 p^{-1/4}\,\mathrm{d}x} \right]^4$$

Let μ_n be the eigenvalues of the "Dirichlet" (EQ)–$(BC)_{DD}$.

Lemma 1. *Let $j_{(MN)}$ be the total of vanishing rows in the matrices $\tilde{M}$ and $\tilde{N}$ $(= 0, 1, 2, 3, 4)$. Then*

$$\underset{\substack{\uparrow \\ n \geqslant 4}}{\mu_{n-j_{(MN)}} \leqslant E_n} \underset{\substack{\uparrow \\ n \geqslant 0}}{\leqslant} \mu_n .$$

<u>Proof</u>. $E_n \leqslant \mu_n$.

$$E_n = \min_{\mathcal{S}} \max_{u \in \mathcal{S},\, \int |u|^2 = 1} (a + c + \tilde{b})(u, u),$$

min over $\mathcal{S} \subset \tilde{\mathcal{D}}_{MN}$ of dim$= n$.

$$\mu_n = \min_{\mathcal{S}_d} \max_{u \in \mathcal{S}_d,\, \int |u|^2 = 1} (a + c)(u, u)$$

min over $\mathcal{S}_d \subset \tilde{\mathcal{D}}_{DD}$ of dim$=n$.

Notice $\tilde{\mathcal{D}}_{DD} \subseteq \tilde{\mathcal{D}}_{MN}$.

Proof. $\mu_{n-j} \leqslant E_n$.

j =total of rows vanishing in $(\tilde{M}, \tilde{N})^t$.

$$E_n = \max_{\mathcal{S}} \min_{u \in \mathcal{S},\, \int |u|^2 = 1} (a + c + \tilde{b})(u, u),$$

max over $\mathcal{S} \subset \tilde{\mathcal{D}}_{MN}$ of $\dim(\tilde{\mathcal{D}}_{MN}/\mathcal{S}) \leqslant n$.

$$\mu_{n-j} = \max_{\mathcal{S}_d} \min_{u \in \mathcal{S}_d,\, \int |u|^2 = 1} (a + c)(u, u)$$

max over $\dim(\tilde{\mathcal{D}}_{DD}/\mathcal{S}_d) \leqslant n - j$.

Since $\dim(\tilde{\mathcal{D}}_{MN}/\tilde{\mathcal{D}}_{DD}) = j$, then $\dim(\tilde{\mathcal{D}}_{MN}/\mathcal{S}_d) \leqslant n$.

8- Upper bound on the number of zeros.

$\#[f]$:=number of vanishing pts. of f in $(0,1)$ count. mult.

Theorem 2. *Let $n \geqslant 4$. If E_n is such that $E_n > q(x)$ for $0 \leqslant x \leqslant 1$, then $\#[y_n] \leqslant n + c_{(MN)}$. Where $c_{(MN)} \leqslant 12$ is a constant.*

Sharp? Probably not: if p, q are constant, $c_{(MN)} \leqslant 3$.

Lemma 1 implies that there is at most 4 eigenvalues below the minimum of $q(x)$.

Sketch of the proof:

- If $(w_n, \mu_n) \in \mathcal{D} \times \mathbb{R}$ is eigenpair of $(EQ) - (BC)_{DD}$, $\#[w_n] = n$.

- Any other solution of (EQ) with $E = \mu_n$ has at most $\#[w_n] + 3$ zeros in $(0,1)$.

- Lemma 1.

- Comparison result of [Leighton, Nehari]: the solutions u, v of $(ru'')'' - Q_1 u = 0$ and $(rv'')'' - Q_2 v = 0$ for $0 < Q_1(x) \leqslant Q_2(x)$ subject to $u(0) = v(0) = u(1) = v(1) = 0$ satisfy $\#[u] \leqslant \#[v] + 3$.

9- Lower bounds?

There is no lower bound for large positive e.v.!

The problem $y^{(4)} - Ey = 0$ with b.c. given by

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 1 & 0 & -1 & b_1 \\ 0 & 1 & b_2 & 1 \end{pmatrix},$$

$b_1 = \frac{u'(1) - u'''(1)}{u(1)}, \qquad b_2 = -\frac{u''(1) + u(1)}{u'(1)},$

$u(x) = \cosh(\beta\pi x/2) - \alpha \cos(\beta\pi x/2) \quad \text{for} \quad \alpha = \cosh(2k\pi).$

$E = (\beta\pi/2)^4$ is e.v. with e.f. $u(x)$ and $\#[u] = 2k + 1$.

Fix $k = 0, 1, 2, \ldots,$ choose β arbitrarily large and recall Lemma 1.

10- Negative eigenvalues.

$(EQ) - (BC)_{MN}$ can have negative eigenvalues with arbitrarily large modulus. There is at most 4 of them.

As $|E|$ increases, the oscillation count increases.

We speculate that the correct asymptotic is

$$\#[y] = (\alpha_p)^{-1/4}|E|^{1/4} + o(|E|^{1/4}), \qquad E \to \infty.$$

for the eigensolution (y, E).

This is certainly the case for constant coefficients.

11- Explanation?

$$E_0 \leqslant \ldots \leqslant E_3 \leqslant E_4 \leqslant \ldots$$

Spectral theorem?