On the spectrum in Smilansky's model of irreversible quantum graphs: the 2-oscillator case

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★ **Graph** $\Gamma = \mathbb{R}, o_1 = 1, o_2 = -1$

★ **The operator** $\mathrm{A}_\alpha, \alpha = (\alpha_+, \alpha_-)$

- **The differential expression**

$$\begin{aligned} \mathcal{A}U = \mathcal{A}_\nu U = -U''_{x^2} \; &+ \; \frac{1}{2}(-U''_{q_+^2} + q_+^2 U) \\ &+ \; \frac{1}{2}(-U''_{q_-^2} + q_-^2 U) \end{aligned}$$

- **Matching conditions**:

$$[U'_x](\pm 1, q_+, q_-) = \alpha_\pm q_\pm U(\pm 1, q_+, q_-).$$

- **The decomposition** $U \sim \{u_{m,n}\}$

$$U(x, q_+, q_-) = \sum_{m,n \in \mathbb{N}_0} u_{m,n}(x) \chi_m(q_+) \chi_n(q_-),$$

$\mathcal{A}U \sim \{L_{m,n} u_{m,n}\} \Rightarrow$

$$(L_{m,n} u)(x) = -u''(x) + r_{m,n} u(x), \qquad x \neq \pm 1; \tag{1}$$

$$r_{m,n} = m + n + 1, \qquad m, n \in \mathbb{N}_0.$$

- **Matching conditions at $x = \pm 1$**

$$[u'_{m,n}](1) = \frac{\alpha_+}{\sqrt{2}}\Big(\sqrt{m+1}u_{m+1,n}(1) + \sqrt{m}u_{m-1,n}(1)\Big)\,; \tag{2}$$

$$[u'_{m,n}](-1) = \frac{\alpha_-}{\sqrt{2}}\Big(\sqrt{n+1}u_{m,n+1}(-1) + \sqrt{n}u_{m,n-1}(-1)\Big) \tag{3}$$

- **The domain $\mathcal{D}_\alpha$ of $\mathbf{A}_\alpha$**

An element $U \sim \{u_{m,n}\}$ lies in $\mathcal{D}_\alpha$ if and only if

1. $u_{m,n} \in H^1(\mathbb{R})$ for all m, n.

2. For all m, n the restriction of $u_{m,n}$ to each interval $(-\infty, -1)$, $(-1, 1)$, $(1, \infty)$ lies in H^2 and moreover,

$$\sum_{m,n} \int_{\mathbb{R}} |L_{m,n} u_{m,n}|^2 dx < \infty.$$

3. The conditions (2) and (3) are satisfied.

★ **Theorem 1**
For all $\alpha_+, \alpha_- \geq 0, \mathbf{A}_\alpha$ is self-adjoint

★ **Theorem 2**

1. $\alpha_\pm < \sqrt{2} \Rightarrow \sigma_{a.c}(\mathbf{A}_\alpha) = [1, \infty)$

2. $\alpha_+ = \sqrt{2}, \alpha_- < \sqrt{2}$(or $\alpha_- = \sqrt{2}, \alpha_+ < \sqrt{2}$) $\Rightarrow \sigma_{a.c}(\mathbf{A}_\alpha) = [1/2, \infty)$

3. $\alpha_+ = \alpha_- = \sqrt{2} \Rightarrow \sigma_{a.c}(\mathbf{A}_\alpha) = [0, \infty)$

4. $\max(\alpha_+, \alpha_-) > \sqrt{2} \Rightarrow \sigma_{a.c}(\mathbf{A}_\alpha) = \mathbb{R}$

★ Theorem 3

1. $\alpha_\pm < \sqrt{2} \Rightarrow \mathbf{A}_\alpha$ is bounded below and its spectrum in $(-\infty, 1)$ is non-empty and finite.

2. Let

$$\Omega_\Psi := \{(x,y) : \Psi(x) \le y \le 1, \quad \Psi(y) \le x \le 1\},$$

where

$$\Psi(t) = e^{-\psi(t)}, \quad \psi(t) = o(t^{-1/4}), t \to 0.$$

Then,

$$\begin{aligned} N_-\left(1; \mathbf{A}_\alpha\right) &\sim \frac{1}{4\sqrt{2}}\sqrt{\frac{\alpha_+}{\sqrt{2}-\alpha_+}} \\ &+ \frac{1}{4\sqrt{2}}\sqrt{\frac{\alpha_-}{\sqrt{2}-\alpha_-}}, \qquad (4) \end{aligned}$$

$$\alpha_\pm \uparrow \sqrt{2}. \qquad (5)$$

uniformly for $(1-\alpha_+/\sqrt{2}, 1-\alpha_-/\sqrt{2}) \in \Omega_\Psi$.

★ **Proof of Theorem 1**

To prove

$$\mathbf{A}_\alpha V = \Lambda V, \;\; (\Lambda \in \mathbb{C}_\pm) \Rightarrow V = 0$$

- $V \sim \{v_{m,n}\}, r_{m,n} = m + n + 1,$

$$v_{m,n}(x) = r_{m,n}^{1/4}\{C_{m,n}^{+}\phi_{m,n}^{+}(x) + C_{m,n}^{-}\phi_{m,n}^{-}(x)\}$$

where $\{\phi_{m,n}^{+}, \phi_{m,n}^{-}\}$ is a basis of $\mathfrak{F} := \{v : -v'' + \zeta^2 v = 0, v \in L^2(\mathbb{R})\}$, with $\zeta^2 = r_{m,n} - \Lambda =: \zeta_{m,n}^2(\Lambda)$.

- $V \in L^2(\mathbb{R}^3) \Leftrightarrow \{C_{m,n}^{+}, C_{m,n}^{-}\} \in \ell^2(\mathbb{N}_0^2; \mathbb{C}^2)$

- Matching conditions (2) and (3) $\Rightarrow$

$$q^{+}_{m+1,n}C^{+}_{m+1,n}$$

$$+\frac{2\mu_{+}p_{m,n}(\Lambda)}{1-e^{-4\zeta_{m,n}(\Lambda)}}\Big(C^{+}_{m,n}-C^{-}_{m,n}e^{-2\zeta_{m,n}(\Lambda)}\Big)$$

$$+q^{+}_{m,n}C^{+}_{m-1,n}=0,$$

$$q^{-}_{m,n+1}C^{-}_{m,n+1}$$

$$+\frac{2\mu_{-}p_{m,n}(\Lambda)}{1-e^{-4\zeta_{m,n}(\Lambda)}}\Big(C^{-}_{m,n}-C^{+}_{m,n}e^{-2\zeta_{m,n}(\Lambda)}\Big)$$

$$+q^{-}_{m,n}C^{-}_{m,n-1}=0$$

where

$$q^{+}_{m,n}=m^{1/2}r^{1/4}_{m,n}r^{1/4}_{m-1,n},$$
$$q^{-}_{m,n}=n^{1/2}r^{1/4}_{m,n}r^{1/4}_{m,n-1};$$
$$p_{m,n}(\Lambda)=\zeta_{m,n}(\Lambda)r^{1/2}_{m,n}.$$

• Let $\mathcal{R}(\Lambda)$ be the infinite matrix which corresponds to this system and $\mathcal{R}'(\Lambda)$ that for the system with exponential terms removed (operators in $\ell^2(\mathbb{N}_0^2;\mathbb{C}^2)$).

• $\mathcal{R}'(\Lambda) = \sum_n{}^{\oplus}\mathcal{J}_n^+(\Lambda) \oplus \sum_m{}^{\oplus}\mathcal{J}_m^-(\Lambda)$
$\mathcal{J}_k^\pm$ Jacobi matrices

$$\begin{aligned} \|\mathcal{J}_k^\pm(i\tau)^{-1}\| &\leq (c\sqrt{|\tau|})^{-1} \\ \Rightarrow \|\mathcal{R}'(i\tau)^{-1}\| &\leq (c\sqrt{|\tau|})^{-1} \end{aligned}$$

• $\mathcal{R}(i\tau) = \mathcal{R}'(i\tau) + \mathcal{N}(i\tau)$
$\mathcal{N}(i\tau)$ block-diagonal, $\|\mathcal{N}(i\tau)\| \leq C(\tau_0), |\tau| \geq \tau_0$.

• For $|\tau|$ large enough, $\mathcal{R}(i\tau)$ has a bounded inverse

$$\Rightarrow V = 0.$$

★ Proof of Theorem 2

• **The operator $\mathbf{A}_\alpha^o$**

$$\mathbf{A}_\alpha^o := \mathbf{A}_{\alpha+}^{+} \oplus \mathbf{A}_{\alpha-}^{-}$$

$$\mathbf{A}_{\alpha_+}^{+} = \sum_{n\in\mathbb{N}_0}^{\oplus} \Big(\mathbf{A}_{\mathbb{R}_+;\alpha_+} + n + 1/2\Big) \quad ,$$
$$\mathbf{A}_{\alpha_-}^{-} = \sum_{m\in\mathbb{N}_0}^{\oplus} \Big(\mathbf{A}_{\mathbb{R}_-;\alpha_-} + m + 1/2\Big) \quad .$$

$\mathbf{A}_{\mathbb{R}_+;\alpha_+}$ one-oscillator operator on $\mathbb{R}_+$ with matching condition at $o_1 = 1$ and Dirichlet condition at 0; similarly for $\mathbf{A}_{\mathbb{R}_-;\alpha_-}$.

• $(\mathbf{A}_\alpha^o - \Lambda)^{-3} - (\mathbf{A}_\alpha - \Lambda)^{-3} \in \mathfrak{S}_1$

• Complete isometric wave operators exist for $(\mathbf{A}_\alpha, \mathbf{A}_\alpha^o)$ and $(\mathbf{A}_\alpha^o, \mathbf{A}_\alpha)$

• Absolutely continuous parts of $\mathbf{A}_\alpha$ and $\mathbf{A}_a^o$ unitarily equivalent $\Rightarrow$Theorem 2.

★ Proof of Theorem 3

- **Quadratic form**

$$\mathbf{a}_\alpha[U] = \mathbf{a}[U] + \alpha_+ \mathbf{b}_+[U] + \alpha_- \mathbf{b}_-[U],$$

where, for $U \sim \{u_{m,n}\}$,

$$\begin{aligned} \mathbf{a}[U] &= \sum_{m,n\in\mathbb{N}_0} \int_{\mathbb{R}} \Big(|u'_{m,n}(x)|^2 + r_{m,n}|u_{m,n}|^2\Big)dx, \\ \mathbf{b}_+[U] &= \mathrm{Re} \sum_{m,n\in\mathbb{N}_0} \sqrt{2m}\, u_{m,n}(1)\overline{u_{m-1,n}(1)}, \\ \mathbf{b}_-[U] &= \mathrm{Re} \sum_{m,n\in\mathbb{N}_0} \sqrt{2n}\, u_{m,n}(-1)\overline{u_{m,n-1}(-1)}. \end{aligned}$$

- $\alpha_\pm \le \sqrt{2} \Rightarrow$

$$|\alpha_+ \mathbf{b}_+[U] + \alpha_- \mathbf{b}_-[U]| \le a[U] + k\|U\|^2_{\mathfrak{H}}$$

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$$\alpha_\pm < \sqrt{2} \Rightarrow \mathbf{a}_\alpha \quad \text{closed}$$

• $\sigma(\mathbf{A}_\alpha) \cap (-\infty, 1)$ **finite and non-empty**

Let $\mathbf{A}_\alpha^{(L)}$ be the operator associated with $\mathbf{a}_\alpha$ on

$\mathbf{d}^{(L)} = \{U \sim \{u_{m,n}\} : u_{m,n}(\pm 1) = 0, m + n \leq L\}$

$\mathbf{A}_\alpha^{(L)} = \mathbf{A}_\alpha^{(L,-)} \oplus \mathbf{A}_\alpha^{(L,+)}$ where

$$\mathbf{A}_\alpha^{(L,-)} = \sum_{m+n \leq L}^{\oplus} \left(-\frac{d^2}{dx^2} + m + n + 1 \right),$$

$$\sigma(\mathbf{A}_\alpha^{(L,-)}) = \sigma_{a.c}(\mathbf{A}_\alpha^{(L,-)}) = [1, \infty)$$

and for any $\lambda_0 > 0$,

$$\mathbf{A}_\alpha^{(L,+)} \geq \lambda_0.$$

Thus

$$\sigma(\mathbf{A}_\alpha^{(L)}) = [1, \infty); \quad \sigma_{a.c}(\mathbf{A}_\alpha^{(L)}) \supseteq [1, \lambda_0)$$

$\mathbf{d}^{(L)}$ has finite co-dimension in $\mathbf{d}$ $\Rightarrow$ resolvents of $\mathbf{A}_\alpha, \mathbf{A}_\alpha^{(L)}$ differ by a finite rank operator.

• **Removing the component** $u_{0,0}$

Let

$$\mathcal{H}^0 := \{U \sim \{u_{m,n}\} : u_{0,0} = 0\}$$

and denote corresponding forms and operators by $\mathbf{a}^0_\alpha, \mathbf{A}^0_\alpha, \cdots$.

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$$0 \le N_-(1; \mathbf{A}_\alpha) - N_-(1; \mathbf{A}^0_\alpha) \le 2$$

• Let $\mathbf{B}_\alpha$ be the bounded operator in d^0 associated with $\alpha_+ b_+ + \alpha_- b_-$ and $\mathcal{F} := \sum^\oplus \mathcal{F}_{\sqrt{m+n}}$. We have $\mathcal{F}^\perp \subset \ker \mathbf{B}_\alpha$ and we can restrict attention to $\mathcal{F}$. Bounded operators $B'_\pm$ are defined on $\ell^2(\mathbb{N}_0^2 \setminus \{0,0\})$ such that

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$$N_-(1; \mathbf{A}^0_\alpha) = N_+(1; -\alpha_+ \mathbf{B}'_+ - \alpha_- \mathbf{B}'_-)$$

•

$$I+\alpha_+B'_+ +\alpha_-B'_- = (I+\alpha_+B''_+)\oplus(I+\alpha_-B''_-)+X_\alpha$$

$$N_-(0; I+\alpha_\pm B''_\pm) \sim \frac{1}{4\sqrt{2}}\sqrt{\frac{\alpha_\pm}{\sqrt{2}-\alpha_\pm}}$$

$$N_+(\varepsilon; |X_\alpha|) \le R\ln^4(K/\varepsilon).$$

References

W.D.Evans and M.Solomyak, Journal of Physics, A: Mathematics and General. **38** (2005), 1-17.

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