Covariant Functional Calculus and Spectrum

Vladimir V. Kisil

1 The traditional approach to functional calculus (F.C.)

Definition 1.1. An analytic functional calculus for an element a of an algebra $\mathfrak{A}$ is a *continuous linear* mapping Φ from an algebra of functions $\mathcal{A}$ to $\mathfrak{A}$ s.t.

1. Φ is a unital algebra homomorphism $\Phi(f \cdot g) = \Phi(f) \cdot \Phi(g)$.
2. There is the initialisation conditions: $\Phi[v_0] = a$ for $v_0(z) = z$.

Covariant Functional Calculus and Spectrum

Vladimir V. Kisil

1 The traditional approach to functional calculus (F.C.)

Definition 1.1. An analytic functional calculus for an element a of an algebra $\mathfrak{A}$ is a *continuous linear* mapping Φ from an algebra of functions $\mathcal{A}$ to $\mathfrak{A}$ s.t.

1. Φ is a unital algebra homomorphism $\Phi(f \cdot g) = \Phi(f) \cdot \Phi(g)$.
2. There is the initialisation conditions: $\Phi[v_0] = a$ for $v_0(z) = z$.

Definition 1.2. A resolvent $R_a(\lambda) = (a - \lambda e)^{-1}$ of element $a \in \mathfrak{A}$ is the image under Φ of the Cauchy kernel $(z - \lambda)^{-1}$.

Spectrum of $a \in \mathfrak{A}$ is the set $\mathbf{sp}\, a$ of all singular points of its resolvent $R_a(\lambda)$.

Spectral Mapping Theorem. $f(\mathbf{sp}\, a) = \mathbf{sp}\, f(a)$ *for an analytic function* f.

Covariant Functional Calculus and Spectrum

Vladimir V. Kisil

1 The traditional approach to functional calculus (F.C.)

Definition 1.1. An analytic functional calculus for an element a of an algebra $\mathfrak{A}$ is a *continuous linear* mapping Φ from an algebra of functions $\mathcal{A}$ to $\mathfrak{A}$ s.t.

1. Φ is a unital algebra homomorphism $\Phi(f \cdot g) = \Phi(f) \cdot \Phi(g)$.
2. There is the initialisation conditions: $\Phi[v_0] = a$ for $v_0(z) = z$.

Definition 1.2. A resolvent $R_a(\lambda) = (a - \lambda e)^{-1}$ of element $a \in \mathfrak{A}$ is the image under Φ of the Cauchy kernel $(z - \lambda)^{-1}$.

Spectrum of $a \in \mathfrak{A}$ is the set $\mathbf{sp}\, a$ of all singular points of its resolvent $R_a(\lambda)$.

Spectral Mapping Theorem. $f(\mathbf{sp}\, a) = \mathbf{sp}\, f(a)$ *for an analytic function* f.

Limits of any F.C. based on an algebra homomorphism:

1. Domain $\mathcal{A}$ should be an algebra, i.e. no H_p, $p < \infty$ or Bergman spaces.
2. Range $\mathfrak{A}$ is not smaller than an algebra generated by a, no refinement.

2 Complex Analysis and Functional Calculus from Groups

Analytic function theory in the unit disk $\mathbb{D}$ is mainly a theory of the *discrete series* representation of $SL_2(\mathbb{R})$ group of 2×2 matrices:

$$\rho_m(g) : f(z) \mapsto \frac{1}{(\alpha - \beta z)^m} f\left(\frac{\bar{\alpha} z - \bar{\beta}}{\alpha - \beta z}\right), \quad g = \begin{pmatrix} \bar{\alpha} & -\bar{\beta} \\ -\beta & \alpha \end{pmatrix} \in SL_2(\mathbb{R}). \tag{2.1}$$

To get a definition of F.C. we replace of a *homomorphism property* by a *symmetric covariance*. One possible realisation discussed here is as follows.

2 Complex Analysis and Functional Calculus from Groups

Analytic function theory in the unit disk $\mathbb{D}$ is mainly a theory of the *discrete series* representation of $SL_2(\mathbb{R})$ group of 2×2 matrices:

$$\rho_m(g) : f(z) \mapsto \frac{1}{(\alpha - \beta z)^m} f\left(\frac{\bar{\alpha} z - \bar{\beta}}{\alpha - \beta z}\right), \quad g = \begin{pmatrix} \bar{\alpha} & -\bar{\beta} \\ -\beta & \alpha \end{pmatrix} \in SL_2(\mathbb{R}). \quad (2.1)$$

To get a definition of F.C. we replace of a *homomorphism property* by a *symmetric covariance*. One possible realisation discussed here is as follows.

Definition 2.1. An analytic functional calculus for an element $a \in \mathfrak{A}$ and an $\mathfrak{A}$-module M is a *continuous linear* mapping $\Phi : A(\mathbb{D}) \to A(\mathbb{D}, M)$ such that

1. Φ is an intertwining operator $\Phi \rho_1 = \rho_a \Phi$ between two representations of the $SL_2(\mathbb{R})$ group ρ_1 (2.1) and ρ_a, where $a \in \mathfrak{A}$ defined bellow.

2 Complex Analysis and Functional Calculus from Groups

Analytic function theory in the unit disk $\mathbb{D}$ is mainly a theory of the *discrete series* representation of $SL_2(\mathbb{R})$ group of 2×2 matrices:

$$\rho_m(g) : f(z) \mapsto \frac{1}{(\alpha - \beta z)^m} f\left(\frac{\bar{\alpha} z - \bar{\beta}}{\alpha - \beta z}\right), \quad g = \begin{pmatrix} \bar{\alpha} & -\bar{\beta} \\ -\beta & \alpha \end{pmatrix} \in SL_2(\mathbb{R}). \quad (2.1)$$

To get a definition of F.C. we replace of a *homomorphism property* by a *symmetric covariance*. One possible realisation discussed here is as follows.

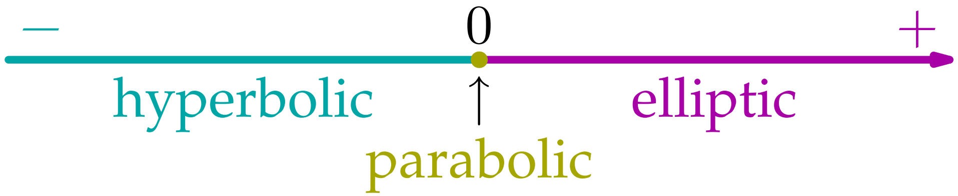
Definition 2.1. An analytic functional calculus for an element $a \in \mathfrak{A}$ and an $\mathfrak{A}$-module M is a *continuous linear* mapping $\Phi : A(\mathbb{D}) \to A(\mathbb{D}, M)$ such that

1. Φ is an intertwining operator $\Phi \rho_1 = \rho_a \Phi$ between two representations of the $SL_2(\mathbb{R})$ group ρ_1 (2.1) and ρ_a, where $a \in \mathfrak{A}$ defined bellow.
2. There is an initialisation condition: $\Phi[v_0] = m$ for $v_0(z) \equiv 1$ and $m \in M$.

A corresponding spectrum of a is the support of the functional calculus Φ.

3 Elliptic, Parabolic, and Hyperbolic Function Theories

Analytic function theories are subject to the following general classification:



We use representations of $SL_2(\mathbb{R})$ group in Clifford valued function spaces. Four dimensional Clifford algebras $\mathcal{C}\ell(a)$ are spanned by 1, e_1, e_2, e_1e_2 s.t.:

$$e_1^2 = -1, \qquad e_2^2 = \begin{cases} -1, & \text{for } \mathcal{C}\ell(e)\text{—elliptic case} \\ 0, & \text{for } \mathcal{C}\ell(p)\text{—parabolic case} \\ 1, & \text{for } \mathcal{C}\ell(h)\text{—hyperbolic case} \end{cases}, \qquad e_1e_2 = -e_2e_1.$$

The subalgebra of $\mathcal{C}\ell(e)$ spanned by 1 and $i = e_1e_2$ is isomorphic (replace!) $\mathbb{C}$. We identify $\mathbb{R}^2$ with the set of vectors $ue_1 + ve_2$ in all $\mathcal{C}\ell(a)$, where $(u, v) \in \mathbb{R}^2$. $SL_2(\mathbb{R})$ consists of 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $ad - bc = 1$ and $a, b, c, d \in \mathbb{R}$.

4 Möbius Transformations of $\mathbb{R}^2$

The same multiplication in $SL_2(\mathbb{R})$ if we replace $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by $\begin{pmatrix} a & -be_1 \\ ce_1 & d \end{pmatrix}$.

For all $\mathcal{C}\ell(a)$ define the Möbius transformation of $\mathbb{R}^2 \to \mathbb{R}^2$ (!) by:

$$\begin{pmatrix} a & -be_1 \\ ce_1 & d \end{pmatrix} : \; ue_1 + ve_2 \;\mapsto\; (ce_1(ue_1 + ve_2) + d)^{-1}(a(ue_1 + ve_2) - be_1).$$

4 Möbius Transformations of $\mathbb{R}^2$

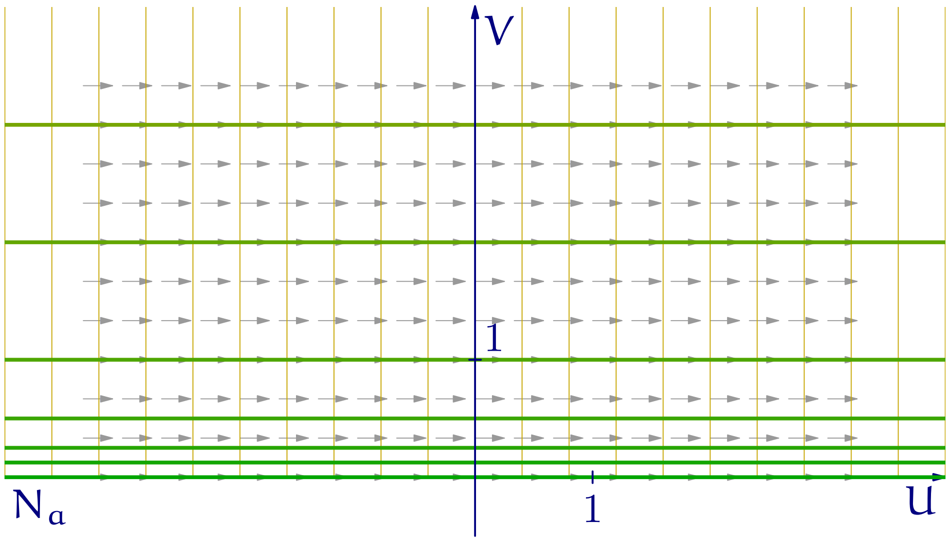
The same multiplication in $SL_2(\mathbb{R})$ if we replace $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by $\begin{pmatrix} a & -be_1 \\ ce_1 & d \end{pmatrix}$.

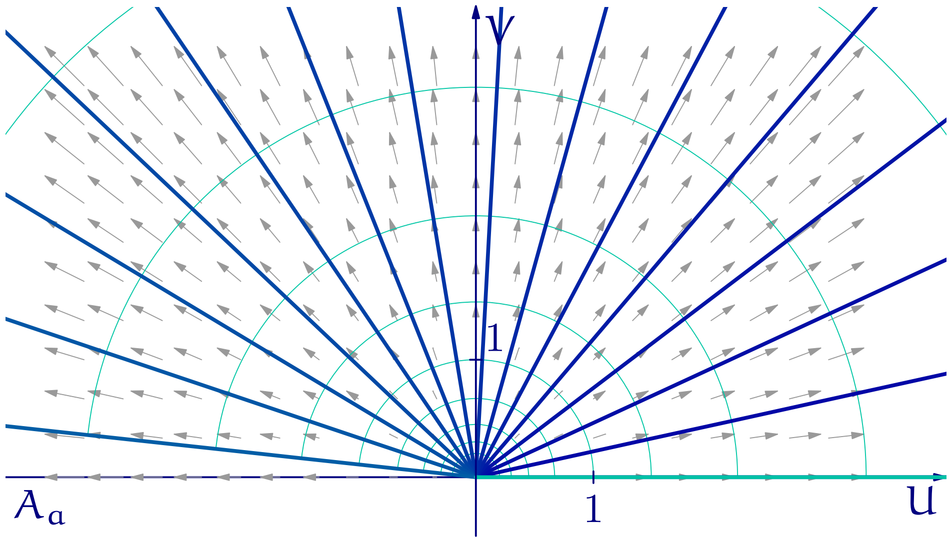
For all $\mathcal{C}\ell(a)$ define the Möbius transformation of $\mathbb{R}^2 \to \mathbb{R}^2$ (!) by:

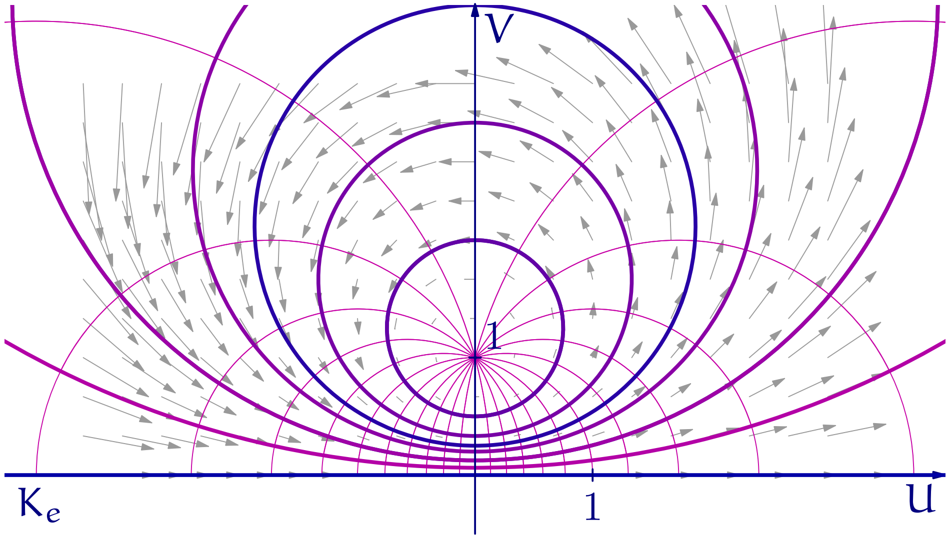
$$\begin{pmatrix} a & -be_1 \\ ce_1 & d \end{pmatrix} : ue_1 + ve_2 \mapsto (ce_1(ue_1 + ve_2) + d)^{-1}(a(ue_1 + ve_2) - be_1).$$

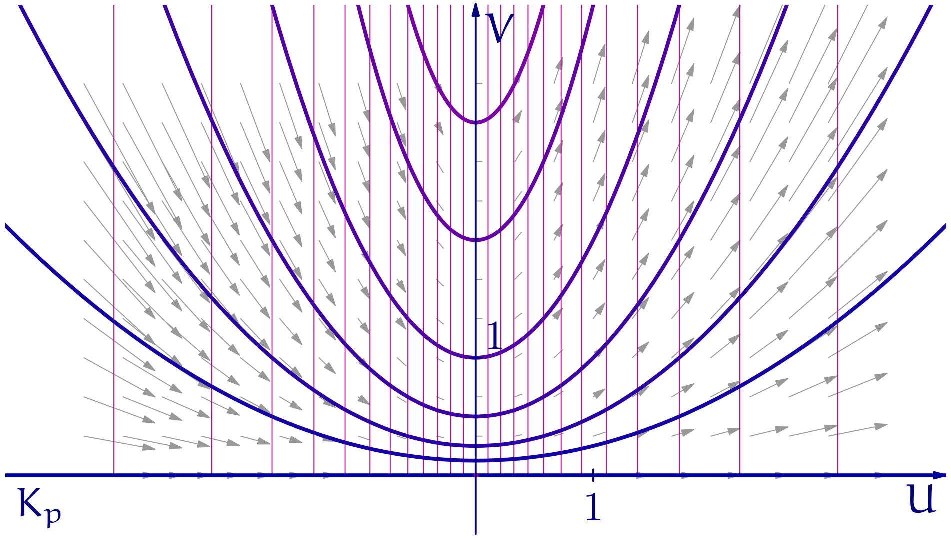
Product $\begin{pmatrix} a & -be_1 \\ ce_1 & d \end{pmatrix} = \begin{pmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{pmatrix} \begin{pmatrix} 1 & xe_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\phi & e_1 \sin\phi \\ e_1 \sin\phi & \cos\phi \end{pmatrix}$ gives

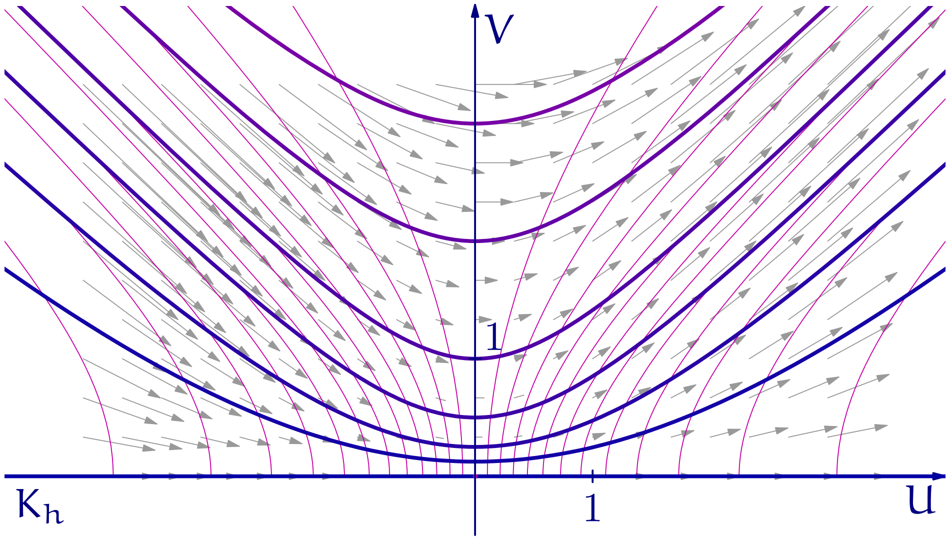
Iwasawa $SL_2(\mathbb{R}) = ANK$. In all $\mathcal{C}\ell(a)$ subgroups A and N acts uniformly:











Vector fields are:

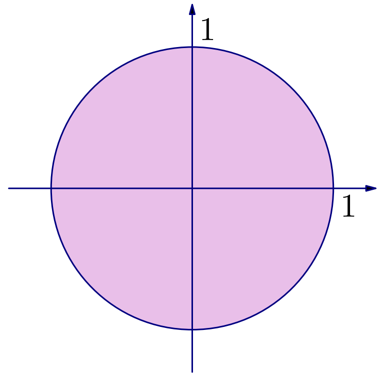
$$\begin{aligned} dK_e(u,v) &= (1+u^2-v^2, \quad 2uv) \\ dK_p(u,v) &= (1+u^2, \quad 2uv) \\ dK_h(u,v) &= (1+u^2+v^2, \quad 2uv) \end{aligned}$$

Figure 1: Depending from $e_2^2 = -1, 0, 1$ the action of subgroup K of $\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$ produces circles, parabolas and hyperbolas.

5 Cayley Transform and Unit "Circles"

The colour code of ANK match to the model, where subgroup is diagonalised.

In elliptic case the standard Cayley transform diagonalise K:



$$\begin{pmatrix}\alpha & \bar{\beta}\\ \beta & \bar{\alpha}\end{pmatrix} = \frac{1}{\sqrt{1-|u|^2}}\begin{pmatrix}e^{i\omega} & 0\\ 0 & e^{-i\omega}\end{pmatrix}\begin{pmatrix}1 & \bar{u}\\ u & 1\end{pmatrix}, \text{ with } \begin{array}{c}\omega = \arg\alpha,\\ u = \beta\bar{\alpha}^{-1},\end{array}$$

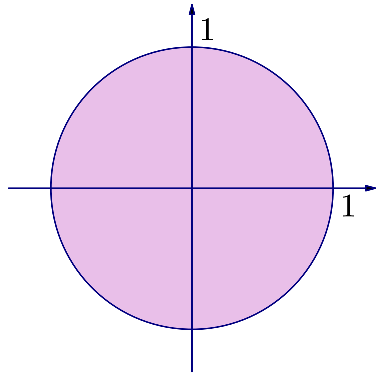
and $|u| < 1$ follows from $|\alpha|^2 - |\beta|^2 = 1$, using notation $i = e_1 e_2$.

5 Cayley Transform and Unit "Circles"

The colour code of ANK match to the model, where subgroup is diagonalised.

In elliptic case the standard Cayley transform diagonalise K:

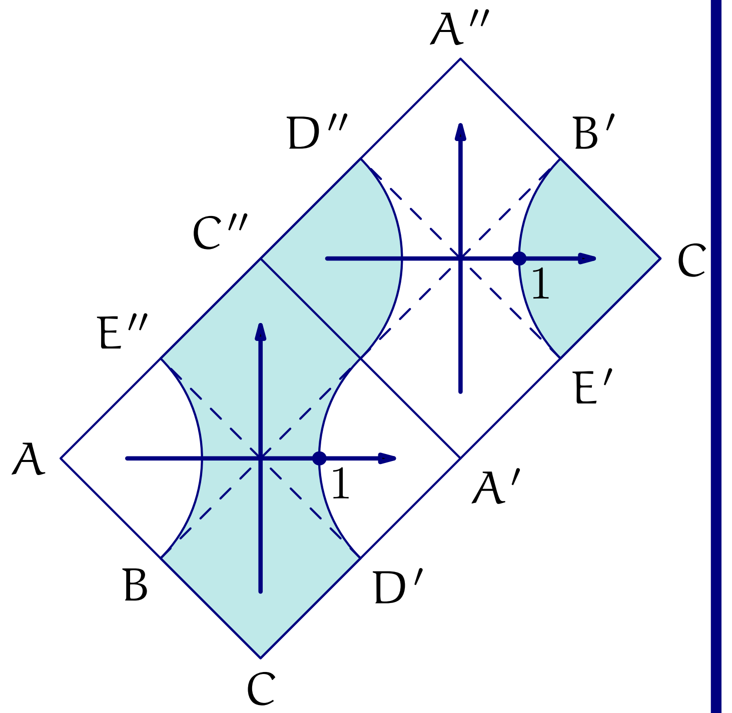
$$\begin{pmatrix}\alpha & \bar\beta\\ \beta & \bar\alpha\end{pmatrix} = \frac{1}{\sqrt{1-|u|^2}}\begin{pmatrix}e^{i\omega} & 0\\ 0 & e^{-i\omega}\end{pmatrix}\begin{pmatrix}1 & \bar u\\ u & 1\end{pmatrix}, \text{ with } \begin{array}{l}\omega = \arg\alpha,\\ u = \beta\bar\alpha^{-1},\end{array}$$



and $|u| < 1$ follows from $|\alpha|^2 - |\beta|^2 = 1$, using notation $i = e_1 e_2$.

In hyperbolic case we analogously diagonalise A:

$$\begin{pmatrix}a & b\\ -b & a\end{pmatrix} = |a|\begin{pmatrix}\frac{a}{|a|} & 0\\ 0 & \frac{a}{|a|}\end{pmatrix}\begin{pmatrix}1 & a^{-1}b\\ -a^{-1}b & 1\end{pmatrix}.$$



However we could not deduce $\left|a^{-1}b\right| < 1$ now!

Geometry: $\mathbb{R}^2$ is not split by the unit circle;

Analysis: Hardy space is not a proper subset of L_2;

Physics: Past and future could be reversed contly.

6 Cauchy and Bergman Integrals as Wavelet Transforms

In the elliptic case Möbius maps give UIR ρ_m from the discrete series of $SL_2(\mathbb{R})$ on Hardy $H_2(\mathbb{T})$ ($=: B_1(\mathbb{D})$) or Bergman $B_m(\mathbb{D})$, $m = 2, 3, \ldots$ spaces:

$$g^{-1} : z \mapsto \frac{\bar{\alpha}z - \bar{\beta}}{\alpha - \beta z}, \qquad \longrightarrow \qquad \rho_m(g) : f(z) \mapsto \frac{1}{(\alpha - \beta z)^m} f\left(\frac{\bar{\alpha}z - \bar{\beta}}{\alpha - \beta z}\right)$$

K-invariant vacuum vector $v_0(z) \equiv 1$ gives wavelets $v_m(g, z) = \rho_m(g)v_0(z)$ ess. depend only from $\bar{u} = \beta\alpha^{-1} \in \mathbb{D}$. Then $v_m(u, z) = (1 - \bar{u}z)^{-m}$ are the Cauchy and Bergman kernels. Thus the universally defined wavelet transforms $\mathcal{W}_m f(u) = \langle f(z), \rho_m v_0(u, z)\rangle$ are Cauchy and Bergman integrals:

$$\mathcal{W}_1 f(u) = \frac{1}{2\pi i} \int_{\mathbb{T}} f(z) \frac{1}{u - z}\, dz, \qquad \mathcal{W}_m f(u) = \int_{\mathbb{D}} f(z) \frac{1}{(1 - u\bar{z})^m} \frac{dz}{(1 - |z|^2)^{m-1}}.$$

6 Cauchy and Bergman Integrals as Wavelet Transforms

In the elliptic case Möbius maps give UIR ρ_m from the discrete series of $SL_2(\mathbb{R})$ on Hardy $H_2(\mathbb{T})$ ($=: B_1(\mathbb{D})$) or Bergman $B_m(\mathbb{D})$, $m = 2, 3, \ldots$ spaces:

$$g^{-1} : z \mapsto \frac{\bar{\alpha} z - \bar{\beta}}{\alpha - \beta z}, \qquad \longrightarrow \qquad \rho_m(g) : f(z) \mapsto \frac{1}{(\alpha - \beta z)^m} f\left(\frac{\bar{\alpha} z - \bar{\beta}}{\alpha - \beta z}\right)$$

K-invariant vacuum vector $v_0(z) \equiv 1$ gives wavelets $v_m(g, z) = \rho_m(g) v_0(z)$ ess. depend only from $\bar{u} = \beta\alpha^{-1} \in \mathbb{D}$. Then $v_m(u, z) = (1 - \bar{u}z)^{-m}$ are the Cauchy and Bergman kernels. Thus the universally defined wavelet transforms $\mathcal{W}_m f(u) = \langle f(z), \rho_m v_0(u, z)\rangle$ are Cauchy and Bergman integrals:

$$\mathcal{W}_1 f(u) = \frac{1}{2\pi i} \int_{\mathbb{T}} f(z) \frac{1}{u - z}\, dz, \qquad \mathcal{W}_m f(u) = \int_{\mathbb{D}} f(z) \frac{1}{(1 - u\bar{z})^m} \frac{dz}{(1 - |z|^2)^{m-1}}.$$

In the hyperbolic case principal series UIR ρ_σ of $SL_2(\mathbb{R})$ produce similarly:

$$[\mathcal{W}_\sigma f](u) = \left|1 + u^2\right|^{1/2} e_{12} \int_{\mathbb{U}} \frac{(-u e_1 z + 1)^\sigma z^\sigma}{(-e_1 u + z)^{1+\sigma}}\, dz\, f(z), \qquad \text{for } \sigma \in \mathbb{R},$$

where $z = e^{e_{12} t}$ and $dz = e_{12} e^{e_{12} t}\, dt$. Again vacuum vector $v_0(z) \equiv 1$ was taken to be A-covariant and wavelet transform is $\mathcal{W}_\sigma f(u) = \langle f(z), \rho_\sigma v_0(u, z)\rangle$.

7 Cauchy-Riemann Equation from Invariant Fields

A $SL_2(\mathbb{R})$-invariant first order diff.op., which annihilate the image of wavelet transform stands for Cauchy-Riemann operator. If $\rho(Y_j)$ is representation of Lie derivative A, N, K without named then C-R operator is given by:

$$D = \rho(Y_1)e_1 + \rho(Y_2)e_2, \qquad \text{and} \qquad \Delta = \rho(Y_1)^2 e_1^2 + \rho(Y_1)^2 e_2^2,$$

its square is the Laplace operator. In elliptic case K is deleted and we get invariant C-R and Laplace operators. In hyperbolic case subgroup A is deleted and formulae produce a type of Dirac and wave operators:

$$D = u_2(e_1\partial_1 + e_2\partial_2), \qquad \text{and} \qquad \Delta = -u_2^2\partial_1^2 + (u_2\partial_2)^2$$

7 Cauchy-Riemann Equation from Invariant Fields

A $SL_2(\mathbb{R})$-invariant first order diff.op., which annihilate the image of wavelet transform stands for Cauchy-Riemann operator. If $\rho(Y_j)$ is representation of Lie derivative A, N, K without named then C-R operator is given by:

$$D = \rho(Y_1)e_1 + \rho(Y_2)e_2, \qquad \text{and} \qquad \Delta = \rho(Y_1)^2 e_1^2 + \rho(Y_1)^2 e_2^2,$$

its square is the Laplace operator. In elliptic case K is deleted and we get invariant C-R and Laplace operators. In hyperbolic case subgroup A is deleted and formulae produce a type of Dirac and wave operators:

$$D = u_2(e_1\partial_1 + e_2\partial_2), \qquad \text{and} \qquad \Delta = -u_2^2\partial_1^2 + (u_2\partial_2)^2$$

8 Taylor Expansion over Eigenfunctions

Vacuum vector v_0 is an eigenfunction of K or A. A wavelet is decomposable over the complete set of its eigenfunctions. The C-R operators kill half of them, only the other half is really needed. In the elliptic case eigenvectors of K are z^m, $m = 0, 1, 2, \ldots$ and the decomposition is the Taylor series: $f(z) = \sum_0^\infty c_n z^n$. In the hyperbolic case eigenvectors of A are z^p, $p \in \mathbb{R}_+$ and a Taylor type expansion is given by the integral $f(z) = \int_0^\infty c(p) z^p \, dp$.

10 Representations of $SL_2(\mathbb{R})$ in Banach Algebras

Let $a \in \mathfrak{A}$ with $\mathbf{sp}\, a \in \bar{\mathbb{D}}$ be fixed in a Banach algebra $\mathfrak{A}$ with the unit e, then

$$g : a \mapsto g \cdot a = (\bar{\alpha} a - \bar{\beta} e)(\alpha e - \beta a)^{-1}, \qquad g \in SL_2(\mathbb{R}) \tag{10.1}$$

is a well defined $SL_2(\mathbb{R})$ action on a subset $\mathbb{A} = \{g \cdot a \mid g \in SL_2(\mathbb{R})\} \in \mathfrak{A}$, i.e. $\mathbb{A}$ is a $SL_2(\mathbb{R})$-homogeneous space. Define resolvent function $R(g, a) : \mathbb{A} \to \mathfrak{A}$:

$$R(g, a) = (\alpha e - \beta a)^{-1} \quad \text{then} \quad R_1(g_1, a) R_1(g_2, g_1^{-1} a) = R_1(g_1 g_2, a). \tag{10.2}$$

10 Representations of $SL_2(\mathbb{R})$ in Banach Algebras

Let $a \in \mathfrak{A}$ with $\mathbf{sp}\, a \in \bar{\mathbb{D}}$ be fixed in a Banach algebra $\mathfrak{A}$ with the unit e, then

$$g : a \mapsto g \cdot a = (\bar{\alpha} a - \bar{\beta} e)(\alpha e - \beta a)^{-1}, \qquad g \in SL_2(\mathbb{R}) \tag{10.1}$$

is a well defined $SL_2(\mathbb{R})$ action on a subset $\mathbb{A} = \{g \cdot a \mid g \in SL_2(\mathbb{R})\} \in \mathfrak{A}$, i.e. $\mathbb{A}$ is a $SL_2(\mathbb{R})$-homogeneous space. Define resolvent function $R(g, a) : \mathbb{A} \to \mathfrak{A}$:

$$R(g, a) = (\alpha e - \beta a)^{-1} \quad \text{then} \quad R_1(g_1, a) R_1(g_2, g_1^{-1} a) = R_1(g_1 g_2, a). \tag{10.2}$$

We could linearise (10.1) in $C(\mathbb{A}, M)$, for a left $\mathfrak{A}$-module M (e.g. $M = \mathfrak{A}$):

$$\rho_a(g_1) : f(g^{-1} \cdot a) \mapsto R(g_1^{-1} g^{-1}, a) f(g_1^{-1} g^{-1} \cdot a) = (\alpha' e - \beta' a)^{-1} f\left(\frac{\bar{\alpha}' \cdot a - \bar{\beta}' e}{\alpha' e - \beta' a}\right).$$

For any $x \in M$ a vacuum vector is $v_x(g^{-1} \cdot a) = x \otimes v_0(g^{-1} \cdot a) \in C(\mathbb{A}, M)$. The wavelet transform associated with v_x is defined by the same formula:

$$\mathcal{W}_m f(g) = \langle f, \rho_a(g) v_x \rangle \quad \text{(an operator version of Cauchy or Bergman integral).}$$

It maps $L_2(\mathbb{A})$ to $C(SL_2(\mathbb{R}), M)$. The Riesz-Dunford calculus is given by

$$\Phi : f \mapsto \mathcal{W}_1 f(0) \qquad \text{for the choice } M = \mathfrak{A} \text{ and } x = e.$$

11 Jet Bundles and Prolongation of ρ_1

Definition 11.1. Two holomorphic functions have nth order contact in a point if their value and their first n derivatives agree at that point.
A point $(z, u^{(n)}) = (z, u, u_1, \ldots, u_n)$ of the jet space $\mathbb{J}^n \sim \mathbb{D} \times \mathbb{C}^n$ is the equivalence class of holomorphic functions having nth contact at the point z.

11 Jet Bundles and Prolongation of ρ_1

Definition 11.1. Two holomorphic functions have nth order contact in a point if their value and their first n derivatives agree at that point.

A point $(z, u^{(n)}) = (z, u, u_1, \ldots, u_n)$ of the jet space $\mathbb{J}^n \sim \mathbb{D} \times \mathbb{C}^n$ is the equivalence class of holomorphic functions having nth contact at the point z.

For a fixed n each holomorphic function $f : \mathbb{D} \to \mathbb{C}$ has nth prolongation (or n-jet) $j_n f : \mathbb{D} \to \mathbb{C}^{n+1}$ defined as follows:

$$j_n f(z) = (f(z), f'(z), \ldots, f^{(n)}(z)).$$

11 Jet Bundles and Prolongation of ρ_1

Definition 11.1. Two holomorphic functions have nth order contact in a point if their value and their first n derivatives agree at that point.

A point $(z, u^{(n)}) = (z, u, u_1, \ldots, u_n)$ of the jet space $\mathbb{J}^n \sim \mathbb{D} \times \mathbb{C}^n$ is the equivalence class of holomorphic functions having nth contact at the point z.

For a fixed n each holomorphic function $f : \mathbb{D} \to \mathbb{C}$ has nth prolongation (or n-jet) $j_n f : \mathbb{D} \to \mathbb{C}^{n+1}$ defined as follows:

$$j_n f(z) = (f(z), f'(z), \ldots, f^{(n)}(z)).$$

The representation ρ_m of the group $SL_2(\mathbb{R})$ in $B_m(\mathbb{D})$ could be prolonged to a representation $\rho_m^{(n)}$ of $SL_2(\mathbb{R})$ by a transformation of the jet space $\mathbb{J}^n$:

$$\rho_m^{(n)}(g) : (z, u, \ldots, u_n) \mapsto (z(g), u(g), \ldots, u_n(g)), \quad \text{where } z(g) = \frac{\bar{\alpha} z - \bar{\beta}}{-\beta z + \alpha},$$

and $u_k(g)$ is the kth derivative of $\rho_m u$ at the point $z(g)$. From the definition: j_n intertwines ρ_1 and $\rho_1^{(n)}$: $\quad j_n \rho_1(g) = \rho_1^{(n)}(g) j_n \quad$ for all $g \in SL_2(\mathbb{R})$.

11 Jet Bundles and Prolongation of ρ_1

Definition 11.1. Two holomorphic functions have nth order contact in a point if their value and their first n derivatives agree at that point. A point $(z, u^{(n)}) = (z, u, u_1, \ldots, u_n)$ of the jet space $\mathbb{J}^n \sim \mathbb{D} \times \mathbb{C}^n$ is the equivalence class of holomorphic functions having nth contact at the point z.

For a fixed n each holomorphic function $f : \mathbb{D} \to \mathbb{C}$ has nth prolongation (or n-jet) $j_n f : \mathbb{D} \to \mathbb{C}^{n+1}$ defined as follows:

$$j_n f(z) = (f(z), f'(z), \ldots, f^{(n)}(z)).$$

The representation ρ_m of the group $SL_2(\mathbb{R})$ in $B_m(\mathbb{D})$ could be prolonged to a representation $\rho_m^{(n)}$ of $SL_2(\mathbb{R})$ by a transformation of the jet space $\mathbb{J}^n$:

$$\rho_m^{(n)}(g) : (z, u, \ldots, u_n) \mapsto (z(g), u(g), \ldots, u_n(g)), \quad \text{where } z(g) = \frac{\bar{\alpha} z - \bar{\beta}}{-\beta z + \alpha},$$

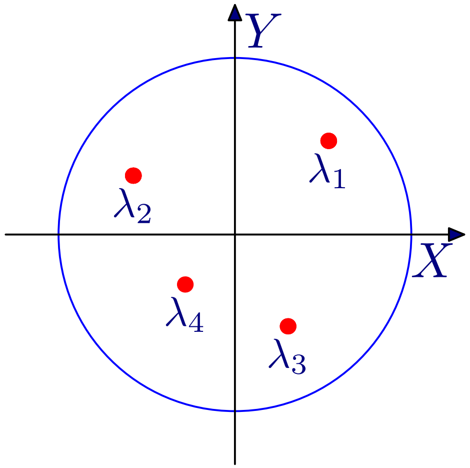
and $u_k(g)$ is the kth derivative of $\rho_m u$ at the point $z(g)$. From the definition: j_n intertwines ρ_1 and $\rho_1^{(n)}$: $\quad j_n \rho_1(g) = \rho_1^{(n)}(g) j_n \quad$ for all $g \in SL_2(\mathbb{R})$.

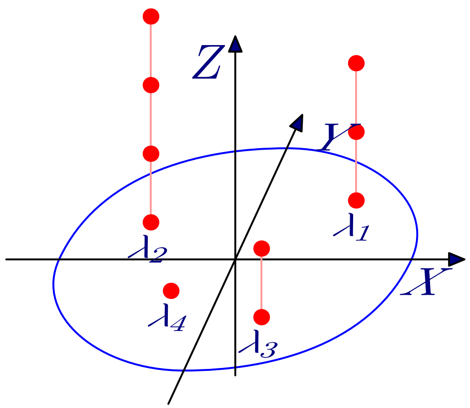
Proposition 11.2. *Let a is a Jordan block of a length k for $\lambda = 0$, and x be its root vector of order k, i.e. $a^{k-1}x \neq a^k x = 0$. Then $\rho_{a,m}$ on v_x is equivalent to ρ_m^k.*

12 Spectrum and Spectral Mapping Theorem

Because of the transitive group of inner automorphisms, which could send any $\lambda \in \mathbb{D}$ to 0, we got the complete characterisation of ρ_a for matrices.

Proposition 12.1 (Jordan normal form). *Representation ρ_a is equivalent to a direct sum of the prolongations $\rho_m^{(k)}$ of ρ_m in the kth jet space $\mathbb{J}^k$ intertwined with inner automorphisms. Consequently the spectrum of a (defined via the functional calculus $\Phi = \mathcal{W}_m$) consists of exactly n pairs (λ_i, k_i), $\lambda_i \in \mathbb{D}$, $k_i \in \mathbb{Z}_+$, $1 \leqslant i \leqslant n$.*





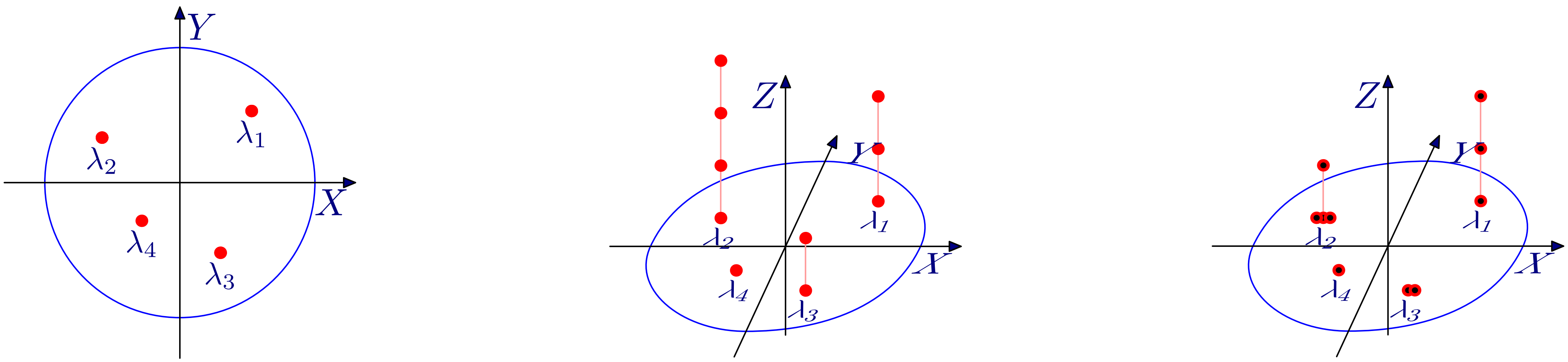
Traditional (left) and new (right) spectra of the matrix: $a = J_3(\lambda_1) \oplus J_4(\lambda_2) \oplus J_2(\lambda_3) \oplus J_1(\lambda_4)$.

Theorem 12.2 (Spectral mapping). *Let* $\phi : \mathbb{D} \to \mathbb{D}$ *be a holomorphic map, let us define* $[\phi_* f](z) = f(\phi(z))$ *and its prolongation* $\phi_*^{(n)}$ *onto the jet space* $\mathbb{J}^n$*. Its associated action* $\rho_1^k \phi_*^{(n)} = \phi_*^{(n)} \rho_1^n$ *on the pairs* (λ, k) *is given by the formula:*

$$\phi_*^{(n)}(\lambda, k) = \left(\phi(\lambda), \left[\frac{k}{\deg_\lambda \phi}\right]\right),$$

where $\deg_\lambda \phi$ *denotes the degree of zero of the function* $\phi(z) - \phi(\lambda)$ *at the point* $z = \lambda$ *and* $[x]$ *denotes the integer part of* x*. Then*

$$\mathbf{sp}\,\phi(a) = \phi_*^{(n)}\,\mathbf{sp}\,a \qquad \text{(which is actually known for Jordan blocks).}$$



14 Calculus of Polynomially Bounded Operators in Bergman Spaces

Standard for a with $\mathbf{sp}\, a \in \bar{\mathbb{D}}$ and $\|a^k\| < Ck^p$ to consider power bounded ra, where $0 < r < 1$, and its H_∞ calculus. A *better regularisation*, $a^k \to a^k/k^p$ rather than $a^k \to r^k a^k$, is achieved in the present framework (although algebra homomorphism is completely lost).

14 Calculus of Polynomially Bounded Operators in Bergman Spaces

Standard for a with $\mathbf{sp}\, a \in \bar{\mathbb{D}}$ and $\|a^k\| < Ck^p$ to consider power bounded ra, where $0 < r < 1$, and its H_∞ calculus. A *better regularisation*, $a^k \to a^k/k^p$ rather than $a^k \to r^k a^k$, is achieved in the present framework (although algebra homomorphism is completely lost).

Since norm of $f(z) = \sum_0^\infty c_k z^k$ in B_m is equivalent to $\sum_0^\infty c_k^2/k^{m-1}$ for polynomially bounded a the resolvent $R(z, a)$ belongs to any B_m with $m > 2(p+1)$. Define a representation of $SL_2(\mathbb{R})$ in $B_m(\mathbb{D} \times \mathbb{A}, M)$ by:

$$\rho'_m : f(u, a) \mapsto \frac{1}{(\bar{\beta}u + \alpha)^{m-1}(\alpha e - \beta a)} f\left(u, \frac{\bar{\alpha}a - \beta e}{\alpha e - \beta a}\right).$$

14 Calculus of Polynomially Bounded Operators in Bergman Spaces

Standard for a with **sp** $a \in \bar{\mathbb{D}}$ and $\|a^k\| < Ck^p$ to consider power bounded ra, where $0 < r < 1$, and its H_∞ calculus. A *better regularisation*, $a^k \to a^k/k^p$ rather than $a^k \to r^k a^k$, is achieved in the present framework (although algebra homomorphism is completely lost).

Since norm of $f(z) = \sum_0^\infty c_k z^k$ in B_m is equivalent to $\sum_0^\infty c_k^2/k^{m-1}$ for polynomially bounded a the resolvent $R(z, a)$ belongs to any B_m with $m > 2(p+1)$. Define a representation of $SL_2(\mathbb{R})$ in $B_m(\mathbb{D} \times \mathbb{A}, M)$ by:

$$\rho'_m : f(u, a) \mapsto \frac{1}{(\bar{\beta}u + \alpha)^{m-1}(\alpha e - \beta a)} f\left(u, \frac{\bar{\alpha}a - \beta e}{\alpha e - \beta a}\right).$$

It is generated by the discrete series representation of $SL_2(\mathbb{R})$ with the lowest weight m. For the vacuum vector $v_0(u, a) \equiv x$ in $B_m(\mathbb{D} \times \mathbb{A}, M)$, where $(x \in M)$, the corresponding functional calculus is given by the integral:

$$f(g \cdot a) = \int_{\mathbb{D}} \frac{f(u)}{(\beta\bar{u} + \bar{\alpha})^{m-1}(\bar{\alpha}e - \bar{\beta}a)} \frac{du}{(1 - |u|^2)^{m-2}}.$$

For Jordan k-blocks with $|\lambda_i| = 1$ it is equivalent to k-prolongation of ρ'_m.

15 Several Variables Spectral Theory

For a joint spectrum of n-tuple of operators we have many alternatives:

- Weyl functional calculus through the Heisenberg group $\mathbb{H}^n$ acting in $L_2(\mathbb{R}^n)$; or Segal-Bargmann type functional calculus through the $\mathbb{H}^n$ acting in $L_2(\mathbb{C}^n)$;

15 Several Variables Spectral Theory

For a joint spectrum of n-tuple of operators we have many alternatives:

- Weyl functional calculus through the Heisenberg group $\mathbb{H}^n$ acting in $L_2(\mathbb{R}^n)$; or Segal-Bargmann type functional calculus through the $\mathbb{H}^n$ acting in $L_2(\mathbb{C}^n)$;
- several complex variables through groups of automorphisms of unit ball or polydisk in $\mathbb{C}^n$. However this is suitable mainly for commuting n-tuples;

15 Several Variables Spectral Theory

For a joint spectrum of n-tuple of operators we have many alternatives:

- Weyl functional calculus through the Heisenberg group $\mathbb{H}^n$ acting in $L_2(\mathbb{R}^n)$; or Segal-Bargmann type functional calculus through the $\mathbb{H}^n$ acting in $L_2(\mathbb{C}^n)$;
- several complex variables through groups of automorphisms of unit ball or polydisk in $\mathbb{C}^n$. However this is suitable mainly for commuting n-tuples;

Or the Clifford analysis through the Möbius group of conformal maps of $\mathbb{C}^n$. The Clifford algebra $\mathcal{C}\ell(n)$ is spanned by $1, e_1, e_2, \ldots, e_n$ with relations

$$e_k^2 = -1 \qquad \text{and} \qquad e_k e_j = -e_j e_k \quad \text{for } k \neq j.$$

15 Several Variables Spectral Theory

For a joint spectrum of n-tuple of operators we have many alternatives:

- Weyl functional calculus through the Heisenberg group $\mathbb{H}^n$ acting in $L_2(\mathbb{R}^n)$; or Segal-Bargmann type functional calculus through the $\mathbb{H}^n$ acting in $L_2(\mathbb{C}^n)$;
- several complex variables through groups of automorphisms of unit ball or polydisk in $\mathbb{C}^n$. However this is suitable mainly for commuting n-tuples;

Or the Clifford analysis through the Möbius group of conformal maps of $\mathbb{C}^n$. The Clifford algebra $\mathcal{C}\ell(n)$ is spanned by $1, e_1, e_2, \ldots, e_n$ with relations

$$e_k^2 = -1 \qquad \text{and} \qquad e_k e_j = -e_j e_k \quad \text{for } k \neq j.$$

Similarly to complex analysis we could derive a Cauchy kernel (cf. resolvent):

$$R(A_1, A_2, \ldots, A_n; \lambda_1, \lambda_2, \ldots, \lambda_n) = \left(\sum_1^n e_k A_k - \sum_1^n e_k \lambda_k I\right)^{-1} \text{ in } B(H) \otimes \mathcal{C}\ell(n).$$

Example 15.1. Let $J_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ be the Pauli matrices. The Cauchy kernel $\frac{-\lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2 e_1 e_2}{(\lambda_1^2 + \lambda_2^2)^2} \begin{pmatrix} (-1-\lambda_1)e_1 - \lambda_2 e_2 & e_2 \\ e_2 & (1-\lambda_1)e_1 - \lambda_2 e_2 \end{pmatrix}$.
Clifford spectrum $\mathbf{sp}_C(J_1, J_2) = \{(0,0)\}$, Weyl spectrum $\mathbf{sp}_W(J_1, J_2) = \mathbb{D}$, Möbius spectrum $\mathbf{sp}_M(J_1, J_2) = \{\rho_1, \rho_1^{(1)}\}$.