An Approximate Spectral Projection of the Laplacian on a Compact Riemannain Manifold

Riemannian (!)

(M, g) - Compact Riemannian manifold.

Define the operator

$$A_\nu = \sqrt{-\Delta + \nu}$$

where $-\Delta$ is the Laplace-Beltrami operator, defined on the space of 1/2-densities, and given in local coordinates by

$$-\Delta u(x) = \frac{-1}{g(x)^{\frac{1}{2}}} \sum_{i,j} \partial_{x^i} \left(g(x) g^{ij}(x) \partial_{x^j} \left(g^{\frac{-1}{2}}(x) u(x) \right) \right),$$

$g(x) = \sqrt{\det \mathrm{g}}$, and ν is a symmetric first-order (pseudo) differential operator.

We construct an approximate spectral projection operator $\chi(\lambda, A_\nu)$ as a pseudodifferential operator, and give an asymptotic expansion of its symbol.

The Approximate Spectral Projection

Let $\Pi(\lambda, s)$ be the indicator function

$$\Pi(\lambda, s) = \begin{cases} 1 & \text{if } s \leq \lambda; \\ 0 & \text{if } s > \lambda, \end{cases}$$

so that $\Pi(\lambda, A_\nu)$ is the spectral projection operator. This is useful for analysing spectral data concerning A_ν. (For example the counting function $N(\lambda) = \#\{\lambda_j \leq \lambda\}$ can be given as $N(\lambda) = \dim(\mathrm{Im}\,(\Pi(\lambda, A_\nu)))$.) To gain smoothness in the λ-variable, we shall consider the following smoothed version of Π. Let $\rho \in (0, 1]$ and define $\chi(\lambda, s) \in C^\infty(\mathbb{R}^2)$ by

$$\chi(\lambda, s) \begin{cases} = 1 & \text{if } s \leq \lambda; \\ \in [0, 1] & \text{if } s \in (\lambda, \lambda + \lambda^\rho]; \\ = 0 & \text{if } s > \lambda. \end{cases}$$

We aim to study the asymptotics of the approximate spectral projection operator $\chi(\lambda, A_\nu)$ as $\lambda \to \infty$. This leads to estimates on the spectral projection operator $\Pi(\lambda, A_\nu)$.

The operator $\chi(\lambda, A_\nu)$ can be defined in an abstract manner using the spectral theorem. Alternatively, we may write it more explicitly in terms of the wave operator:

$$\chi(\lambda, A_\nu) = \int \hat{\chi}(\lambda, t) e^{itA_\nu} \bar{d}t$$

where $\hat{}$ denotes Fourier transform with respect to the second variable. The wave operator e^{itA_ν} is not a pseudodifferential operator and so to avoid its use we write

$$\chi(\lambda, A_\nu) = \int \widehat{\chi_\nu}(\lambda, t) e^{itA_\nu^\mu} \bar{d}t \qquad (1)$$

for some $\mu \in (0, 1)$, where $\chi_\mu(\lambda, s) = \chi(\lambda, s^{1/\mu})$.

The Technical Bit

We analyse (1) in two parts. First, it is shown that for t away from the origin, this operator has a smooth Schwartz kernel that is rapidly decreasing with respect to λ. The only interesting spectral data is therefore to be found when $|t|$ is small.

The symbol of $\chi(\lambda, A_\nu)$ is thus written as

$$\begin{aligned}
\sigma_{\chi(\lambda,A_\nu)}(\lambda;x,\xi) &= \int \widehat{\chi_\mu}(\lambda,t)e^{it|\xi|_x^\mu}\,đt \\
&+ \sum_{j=1}^{N} b_{j,\nu}(x,\xi)\int \widehat{\chi_\mu}(\lambda,t)(it)^j\,đt \\
&+ \int \widehat{\chi_\mu}(\lambda,t)e^{it|\xi|_x^\mu}(it)^{N+1}R_{N+1}(t;x,\xi)\,đt \\
&= \chi(\lambda,|\xi|_x) \\
&+ \sum_{j=1}^{N} b_{j,\nu}(x,\xi)\partial_r^j\chi_\mu(\lambda,r)|_{r=|\xi|_x^\mu} \\
&+ \int \widehat{\chi_\mu}(\lambda,t)e^{it|\xi|_x^\mu}(it)^{N+1}R_{N+1}(t;x,\xi)\,đt.
\end{aligned}$$

The first two terms give an expansion

$$\chi(\lambda,|\xi|_x) + \sum_{j=1}^{N} c_{j,\nu}(x,\xi)\chi^{(j)}(\lambda,|\xi|_x)$$

and these terms become 'better behaved' in the sense that

$$|\chi^{(j)}(\lambda,|\xi|_x)| \leq c_j'(1+\lambda+|\xi|_x)^{-j\rho},$$

whilst the functions $c_{j,\nu}$ are bounded.

It is shown that the operator $U_\mu(t) = e^{itA_\nu^\mu}$ can be written as a pseudodifferential operator

$$U_\mu(t)u(x) = \int g(x)^{\frac{1}{2}} g(y)^{\frac{-1}{2}} e^{i\phi(x,\xi;y)} \sigma_{U_\mu(t)}(t;x,\xi) u(y) dx d\!\!{}^-\xi$$

where $\phi(x,\xi;y)$ is the *phase function* on T^*M defined by

$$\phi(x,\xi;y) = -\langle \dot\gamma_{y,x}(0), \xi \rangle,$$

where $\gamma_{y,x}(\tau)$ is the unique geodesic joining $x = \gamma_{y,x}(0)$ to (nearby) $y = \gamma_{y,x}(1)$ and $\sigma_{U_\mu(t)}(t;x,\xi)$ is the *symbol* of $U_\mu(t)$. For any $N \in \mathbb{N}$ this function can be expanded as

$$\sigma_{U_\mu(t)}(t;x,\xi) = e^{it|\xi|_x^\mu} \left[1 + \sum_{j=1}^{N} (it)^j b_{j,\nu}(x,\xi) + (it)^{N+1} R_N(t;x,\xi) \right]$$

where $|b_{j,\nu}(x,\xi)| \le c_j (1 + |\xi|_x)^{-j(1-\mu)}$ and

$$|R_N(t;x,\xi)| \le C_{N+1} (1 + |\xi|_x)^{-(N+1)(1-\mu)},$$

where C_{N+1} is independent of $t \in [-\varepsilon, \varepsilon]$.

Estimates on $\widehat{\chi_\mu}(\lambda, t)$ show that decay in λ is related to the order of zero at $t = 0$. Therefore the remainder term can be estimated by a multiple of $(1 + \lambda + |\xi|_x)^{-(N+1)(1-\mu)}$. Letting $N \to \infty$ therefore gives us the following

Theorem The approximate spectral projection operator $\chi(\lambda, A_\nu)$ can be written as

$$\chi(\lambda, A_\nu)u(x) = \int g(x)^{\frac{1}{2}} g(y)^{\frac{-1}{2}} e^{i\varphi(x,\xi;y)} \sigma_{A_\nu}(\lambda; x, \xi) u(y)\, dy\, đ\xi$$

where $\sigma_{\chi(\lambda,A_\nu)}(\lambda; x, \xi)$ has the expansion

$$\chi(\lambda, |\xi|_x) + \sum_{j=1}^{\infty} c_{j,\nu}(x, \xi)\chi^{(j)}(\lambda, |\xi|_x)$$

for some bounded functions $c_{j,\nu}$.

Applications

It is straightforward to see that

$$\chi(\lambda, A_\nu) \leq \Pi(\lambda + \lambda^\rho, A_\nu) < \chi(\lambda + \lambda^\rho, A_\nu). \quad (2)$$

For a bounded pseudodifferential operator B, define the generalised counting function by

$$N_B(\lambda) = \text{trace}(B^* \Pi(\lambda, A_\nu) B)$$

and

$$\tilde{N}_B(\lambda) = \text{trace}(B^* \chi(\lambda, A_\nu) B)$$

when such functions are finite. From (2) it follows that

$$\tilde{N}_B(\lambda) \leq N_B(\lambda + \lambda^\rho) \leq \tilde{N}_B(\lambda + \lambda^\rho).$$

The function $\tilde{N}_B(\lambda)$ can be written in terms of the symbol:

$$\tilde{N}_B(\lambda) = \int \sigma_{B^* \chi(\lambda, A_\nu) B}(\lambda; x, \xi) dx d\xi$$

and so our estimates on the symbol of the approximate spectral projection can yield estimates on $N_B(\lambda)$.

Taking $B = I$, then $N_I(\lambda)$ is the counting function, and this approach yields the estimate

$$N(\lambda) = \frac{\text{vol}(M)}{(4\pi)^{n/2}\Gamma(\frac{n}{2}+1)} \lambda^n + O(\lambda^{n-1+\rho}).$$

Although this result is not as strong as the optimal remainder of $O(\lambda^{n-1})$ gained by analysing the wave kernel (Levitan, Hörmander, ...), it is to be hoped that this approach could lead to interesting results concerning, for example, quantum limits of eigenfunctions.

Another interesting question to ask is to what extent the metric g and the perturbation ν affect the span of $\{\varphi_j : \lambda_j \leq \lambda\}$ where φ_j are the eigenfunctions of A_ν. An analysis of $\Pi(\lambda, A_\nu)(I - \Pi(\lambda, A_\eta))$, approximated in terms of χ may yield estimates.

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