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Sharp version of the Birman-Solomyak theorem on estimates of entropy numbers

Def. Let (X, ρ) be a metric space, $A \subset X$, $n \in \{1, 2, 3, \dots\}$. Denote by $e_n(A)$ (the n^{th} entropy number of A) the infimum of all those $\varepsilon > 0$ such that there are 2^{n-1} balls in X of radius $\varepsilon > 0$ which cover A

Def. Let X, Y be Banach spaces, $T: X \to Y$ be a l.b. operator, $n \in \mathbb{N}$. Denote by $e_n(T) \overset{\text{def}}{=} e_n(T(B_X))$, where B_X is the unit ball in X.

$d \in \mathbb{N}$, $\quad \ell > d\left(\frac{1}{p} - \frac{1}{q}\right)$. Then

$\exists \quad c_1(\ell, d) > 0, \quad C_2(\ell, d, p, q)$

such that $\quad \forall \quad n \in \{1,2,3,\dots\}$

$$c_1(\ell,d)\, n^{-\frac{\ell}{d}} \le \overbrace{e_n\left(id : W_p^{\ell}(Q) \hookrightarrow L_q(Q)\right)}^{***} \le C_2(\ell,d,p,q)\, n^{-\ell/d}.$$

i) $\ell p > d$

$$*** \le C_2(\ell, d, p)\, n^{-\ell/d}$$

ii) $\ell p \le d \qquad q < p_* = p_*(\ell, d, p)$

$$\boxed{\frac{\ell}{d} = \frac{1}{p} - \frac{1}{p^*}}$$

$$*** \le C_2(\ell, d, p, q)\, n^{-\ell/d}$$

Th.1 Let $1 < p < \infty$, $\ell p < d$, $q = p_*$,

$a - b \geq \frac{\ell}{d}$. Then $\exists$

$c_1(\ell, d)$, $C_2(\ell, d, p, a)$ such that

$\forall n \in \{1, 2, 3, \dots\}$

$$c_1 n^{-\frac{\ell}{d}} \leq e_n\left(id : W^\ell_p \mathrm{Log}_a(Q) \to L_q \mathrm{Log}_b(Q) \right)$$
$$\leq C_2 n^{-\frac{\ell}{d}}$$

$a = 0,\ b = -\frac{\ell}{d}$ or $a = \frac{\ell}{d},\ b = 0$

$a - b \geq \frac{2\ell}{d}$ (E-T) 1992

$a - b > \frac{\ell}{d}$ (E-N 1997)

$$c_1(\ell, d)\, n^{-\frac{\ell}{d}} \leq e_n\left(id : W^\ell_{H_1}(Q) = H^\ell_{1,*}(Q) \hookrightarrow L_q(Q) \right) \leq$$
$$\leq C_2(\ell, d)\, n^{-\frac{\ell}{d}}, \quad \ell = d(1 - 1/q) \in \mathbb{N}, \quad \ell < d$$

$c_1(\ell, d)\, n^{-\ell/d} \leq e_n (id :$

$b \leq -\frac{\ell}{d}$ Then $\exists\ c_1(\ell,d),\ c_2(\ell,d)$

$\forall n \in \mathbb{N}$

$$c_1 n^{-\ell/d} \leq e_n\left(F^{\ell}_{1,2}(Q) = W^{\ell}_{H_1}(Q) \to L_q \operatorname{Log}_b\right) \leq C_2\, n^{-\ell/d}$$

2°) Let $\ell \in \mathbb{N}$, $\ell < d$, $\ell/d = 1/p$, $a \geq \frac{\ell}{d}$

Then $\exists\ c_1(\ell,d),\ C_2(\ell,d)\quad \forall n \in \mathbb{N}$

$$c_1 n^{-\ell/d} \leq e_n\left(id : W^{\ell}_{p} \operatorname{Log}_a \longrightarrow BMO = F^{0}_{\infty,2}\right) \leq C_2\, n^{-\ell/d}$$

St. Let $1 < p, q < \infty$, $\ell p < d$, $q = p^*$, $\ell \in \mathbb{N}$ and let E be a r.i. Banach space. Suppose that $\exists$, C_2 such that

$$e_n\left(id : W^{\ell}_{E}(Q) \hookrightarrow L_q(Q)\right) \leq C_2 n^{-\ell/d}$$

$\forall n = 1, 2, \ldots$. Then $E \subset L_p \operatorname{Log}_a(Q)$,

$a = -\ell/d$.

$X(\ell, p)$ if $\varepsilon < p < \bar{\varepsilon}$ then $\exists$

$c_1(\varepsilon, \ell, d)$, $c_2(\varepsilon, \ell, d)$ such that

$$c_1(\varepsilon, \ell, d)\, n^{-\frac{\ell}{d}} \leq e_n(id : X(\ell, p) \to L_p) \leqslant c_2(\varepsilon, \ell, d)\, n^{-\frac{\ell}{d}}$$

Let φ be a „good" function ; $\varphi \in C_0^\infty$

Then $\exists$ $c(\varphi, \ell, d)$ $\forall f \in X(\ell, p)$ $\forall m \in N$

$$\exists\ \Gamma_m \subset N \times \mathbb{Z}^n \quad \exists\ f_{\Gamma_m} = \sum_{(i,k) \subset \Gamma_m} \alpha_{i,k}\, \varphi_{i,k}$$

$$\varphi_{i,k}(x) = \varphi\left(\frac{x - 2^{-i} k}{2^{-i}} \right) ,$$

$$\| f - f_{\Gamma_m} \|_{L_p(Q)} \leqslant 2^{-m\ell} \, \| f \|_X \; c(\varphi, \ell, d)$$

$$\sum_{s=1}^{s=+\infty} \# \{ (i,k) \in \Gamma \mid m + 2^s \leq i \leq m + 2^{s+1} \} \, 2^{(s-m)d} \leqslant$$

$$\leqslant 1 .$$

Th. 2 Let $\varepsilon > 0$, $\ell \in \mathbb{N}$, $d \in \mathbb{N}$. Then

$\exists\ c_1(\varepsilon, \ell, d)$, $C_2(\varepsilon, \ell, d)$

such that $\forall\ 1 \leq p, q \leq \infty$

$1+\varepsilon < p \leq q < \infty$, $\ell p < \frac{1}{\varepsilon}$ $\forall\ n \in \mathbb{N}$

$$c_1 \min\left(1, \left(\frac{1}{\left(\frac{1}{q} - \frac{1}{p_*}\right) n}\right)^{\frac{\ell}{d}}\right) \leq$$

$$\leq e_n\left(\mathrm{id}: W_p^\ell(Q) \to L_q(Q)\right) \leq$$

$$\leq C_2 \min\left(1, \frac{1}{\left(\frac{1}{q} - \frac{1}{p_*}\right) n}\right)^{\ell/d}$$

1° ? $1+\varepsilon < p \Rightarrow 1 \leq p$

2° Let $\ell \in \mathbb{N}$, $d \in \mathbb{N}$, $\ell < d$. Then $\exists$

$c_1(\ell, d)$ such that $\forall\ p \geq 1$, $p\ell > d$,

$$e_n\left(W_p^\ell(Q) \to L_\infty(Q)\right) \geq$$

$$\geq \frac{c}{n^{\ell/d}} \left(\frac{1}{p} - \frac{\ell}{d}\right)^{-1} \quad n \in \mathbb{N} \quad n \gg 1$$