A. A. Shkalikov (Moscow State Univ.)

Invariant subspaces of dissipative operators in Krein spaces

Let H be a separable Hilbert space and $J = P_+ - P_-$ be a canonical symmetry ($J^2 = P_+ + P_- = 1$).

$\mathcal{K} = \{H, J\}$ equiped with indefinite inner product

$$[x, y] = (Jx, y), \quad x, y \in H$$

is called Krein space (or Pontrjagin space $\Pi_\varkappa = \{H, J\}$ if rank $P_+ = \varkappa < \infty$).

Def. A subspace $\mathcal{L}$ is nonnegative in $\mathcal{K}$ if $[x, x] \geq 0 \quad \forall x \in \mathcal{L}$.

It is maximal nonnegative if there are no proper extensions of $\mathcal{L}$.

Def. An operator A is *dissipative* in H if

$$\operatorname{Im}(Ax, x) \geq 0 \quad \forall x \in \mathcal{D}(A).$$

It is max. dissipative if there are no proper (dissipative) extensions of A ($\Leftrightarrow \mathbb{C}^- \subset \rho(A)$, where $\mathbb{C}^-$ is open lower-half plane).

Def. A is dissipative in Krein space $\mathcal{K} = \{H, J\}$ if JA is dissipative in H. A is m-dissipatdve in $\mathcal{K}$ if JA is m-dissipative in H.

Symmetric and selfadjoint operators in $\mathcal{K}$ are defined analogously.

Let $H = H_+ \oplus H_-$, $H_\pm = P_\pm(H)$, $\mathcal{D}_\pm = \mathcal{D}(A) \cap H_\pm$.

Assumption: $\mathcal{D}(A) = \mathcal{D}_+ \oplus \mathcal{D}_-$ (it is sufficient to assume that $\mathcal{D}_+ \oplus \mathcal{D}_-$ is a core of A) $\Leftrightarrow$ A admits matrix representation

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} P_+AP_+ & P_+AP_- \\ P_-AP_+ & P_-AP_- \end{pmatrix},$$

where $x = x_+ + x_-$ are idetified with colums $x = \begin{pmatrix} x_+ \\ x_- \end{pmatrix}$.

Background:

Th. (Sobolev, 1941, 1962). A selfadjoint operator in Π_1 has at least one eigenvector corresponding to an eigenvalue $\lambda \in \overline{\mathbb{C}}^+$.

Th. (Pontrjagin, 1944). Let A be selfadjoint in $\Pi_\varkappa$, $\varkappa < \infty$. Then

(a) $\exists$ max. nonnegative subspace $\mathfrak{L}^+$ invariant with respect to A;

(b) among these subspaces $\exists$ $\mathfrak{L}^+$ such that
$$\sigma(A^+) \subset \overline{\mathbb{C}}^+, \quad A^+ = A/_{\mathfrak{L}^+}.$$

Th. (Langer, 1961). Let A be selfadjoint in $\mathcal{K}$ and

(i) $\mathcal{D}(A) \supset H_+$ ($\Leftrightarrow A_{11}$ and A_{21} are bounded)

(ii) A_{12} is compact.

Then (a) & (b) hold.

Th. (Krein, 1948, 1964). Analogues of Pontrjagin and Langer theorems are true for unitary operators in $\Pi_\varkappa$ and $\mathcal{K}$, respectively.

M. Krein proposed a shorther elegant approach to prove (a) by means of Shauder-Tikhonov fixed point theorem.

Th. (Krein and Langer, 1971, Azizov 1972). Let A be m-dissipative in $\Pi_\varkappa$. Then (a) and (b) hold.

Th. (Azizov, Khoroshavin 1981). Let A be a contraction in Krein space and A_{12} be compact. Then (a) & (b) hold if $\mathbb{C}^-$ is replaced by the open unit disk.

Th. (Azizov, 1985). Analogue of the previous result holds for m-dissipative operators in $\mathcal{K}$ provided that $\mathcal{D}(A) \supset H_+$ and A_{12} is A_{22}-compact.

Th. (Shkalikov, 2004). Let

(i) A be dissipative in $\mathcal{K}$;

(ii) A_{22} be m-dissipative in H_- ($\Leftrightarrow \exists\ (A_{22}-\mu)^{-1}$ for some $\mu \in \mathbb{C}^-$);

(iii) $F(\mu) := (A_{22}-\mu)^{-1} A_{21}$ be bounded;

(iv) $G(\mu) := A_{12}(A_{22}-\mu)^{-1}$ be compact;

(v) $S(\mu) := A_{11} - A_{12}(A_{22}-\mu)^{-1} A_{21}$ be bounded.

Then (a) and (b) hold.

Theorem is that firstly the Langer condition $\mathcal{D}(A) \supset H_+$ was droped out. In particular, for a model matix operator

$$A = \begin{pmatrix} u(x) & \frac{d}{dx} \\ \frac{d}{dx} & \frac{d^2}{dx^2} \end{pmatrix}, \qquad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which is selfadjoint in $\mathcal{K} = \{H, J\}$, $H = L_2(0,1) \times L_2(0,1)$, provided that the domain of A is choosen properly, one can guarantee the validity of properties (a) and (b).

The main ~~goal~~ goal of this talk is to prove (a) and (b) provided that only assumptions (i) – (iv) are valid.

It turns out that we need no assumptions for the transfer function $S(\mu)$.

New problems arise if we start working with unbounded entries and reject Langer condition $\mathcal{D}(A) \supset H_+$. In this case, if we succeed to prove (a) & (b), we come to the following interesting problems

(c) does the operator $A^+ = A/_{\mathcal{L}^+}$ generate a C_0-semigroup, or holomorphic semigroup?

We shall provide some sufficient conditions for positive answer to this question.

A subspace $\mathcal{L}$ is A-invariant in classical sense if $\mathcal{L} \subset \mathcal{D}(A)$ and $A(\mathcal{L}) \subset \mathcal{L}$. We accept the following

<u>Def.</u> $\mathcal{L}$ is A-invariant if $\mathcal{D}(A) \cap \mathcal{L}$ is dense in $\mathcal{L}$ and $Ax \in \mathcal{L}$ for all $x \in \mathcal{D}(A) \cap \mathcal{L}$.

Let us formulate the main results.

<u>Theorem</u> A. Conditions (i)-(iv) imply (a).

<u>Theorem</u> B. Property (b) holds if and only if assumption (i) is replaced by

(i') A is m-dissipative in $\mathcal{K}$.

For convenience we accept

Def. B is a generator of H_0-semigroup if $\forall\, \varepsilon > 0$ $B - \varepsilon$ generates a holomorphic semigroup.

Theorem C. iA^+ generates a C_0-semigroup of exponential type 0 is one of the following conditions holds

(1) A_{12} is compact

(2) $-iA_{22}$ generates an H_0-semigroup.

Theorem D. iA^+ generates an exponentially stable semigroup if either (1) or (2) holds and A is uniformly dissipative in $\mathcal{K}$.

Theorem E. There is $\mu \in \mathbb{C}^+$ such that $iS(\mu)$ generates an H_0-semigroup. Then iA^+ generates H_0-semigroup.

The main lines of the proof of the first two theorems.

Assumptions (ii)–(iv) allow to use Frobenious–Shur factorization

$$A-\mu=\begin{pmatrix}1 & G\\ 0 & 1\end{pmatrix}\begin{pmatrix}S-\mu & 0\\ 0 & A_{22}-\mu\end{pmatrix}\begin{pmatrix}1 & 0\\ F & 1\end{pmatrix}$$

where $G=G(\mu)$, $F=F(\mu)$ and $S=S(\mu)$ is the transfer function defined on the domain $\mathcal{D}(S)=\mathcal{D}_+$.

Lemma 1

$$JA+\mu=J\begin{pmatrix}1 & G\\ 0 & 1\end{pmatrix}\begin{pmatrix}S+\mu & 0\\ 0 & A_{22}-\mu\end{pmatrix}\begin{pmatrix}1 & 0\\ F & 1\end{pmatrix}$$

Proof by direct verification.

Lemma 2 $\forall \mu\in\mathbb{C}^+$ and $\forall x\in\mathcal{D}_+$ we have

$$(Sx_+,x_+)=\left(JA\begin{pmatrix}x_+\\ -Fx_+\end{pmatrix},\begin{pmatrix}x_+\\ -Fx_+\end{pmatrix}\right)+$$

$$+\mu(Fx_+,Fx_+).$$

Proof by direct verification.

Corollary (important). $S = S(\mu)$ with domain $\mathcal{D}(S) = \mathcal{D}_+$ is dissipative in H_+ provided that assumption (i) holds. Also, S is closable. The closure of S is m-dissipative in H_+. $\Longleftrightarrow$ A is m-dissipative in $\mathcal{K}$.

Lemma 3 (important). Let a subspace $\mathcal{L}$ have a representation of the form

$$\mathcal{L} = \left\{ x : \ x = \begin{pmatrix} x_+ \\ Kx_+ \end{pmatrix}, \ x_+ \in H_+ \right\}$$

where $K : H_+ \to H_-$ is a bounded operator. Then $\mathcal{L}$ is A-invariant $\Longleftrightarrow$

$$(1 - KG)(A_{22} - \mu)(F + K) = K(S - \mu)$$

(the so-called Riccati equation for K).

Proof. For $x_+ \in \mathcal{D}_+$

$$(A - \mu)\begin{pmatrix} x_+ \\ Kx_+ \end{pmatrix} = \begin{pmatrix} (S-\mu)x_+ + G(A_{22}-\mu)(F+K)x_+ \\ (A_{22}-\mu)(F+K)x_+ \end{pmatrix}.$$

Assuming that $\mathcal{L}$ is A-invariant we find $y_+ \in H_+$ such that

$$[(S-\mu)+G(A_{22}-\mu)(F+K)]\,x_+ = y_+,$$
$$(A_{22}-\mu)(F+K)\,x_+ = K y_+ .$$

Substituting the first equality in the second one we come to Riccati equation for K.

Conversely, Riccati equation for K implies the last two equations with some y_+, therefore the graph subspace $\mathcal{L}$ is A-invariant. $\square$

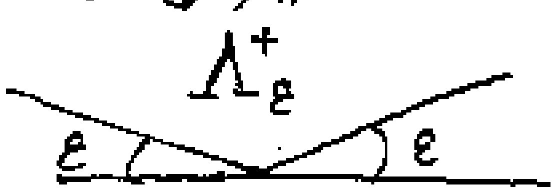
<u>Remark.</u> Pontrjgin used:

$\mathcal{L}$ is A-invariant $\Longleftrightarrow$

$$A_{21}+A_{22}K-KA_{11}-KA_{12}K=0$$

However this form of Riccati equation is unconvenient while working with unbounded entries A_{ij}.

<u>Lemma</u> 4. Assume that $G(\mu)$ is compact for some $\mu\in\mathbb{C}^+$. Then it is compact for all $\mu\in\mathbb{C}^+$ and $\|G(\mu)\|\to 0$ as $\mu\to\infty$ and $\mu\in\Lambda_\varepsilon^+$.



<u>Proof</u> is sumple.

<u>Lemma</u> 5. A subspace $\mathcal{L}$ is max nonnegative $\Longleftrightarrow$ $\mathcal{L}$ has graph representation

$$\mathcal{L} = \left\{ x = \begin{pmatrix} x_+ \\ Kx_+ \end{pmatrix},\ x_+ \in H_+ \right\}$$

with the angle operator K, $\|K\| \leq 1$.

<u>Corollary</u>. Take $\mu \in \mathbb{C}^+$ such that $\|G(\mu)\| < 1/2$. Then (a) holds $\Longleftrightarrow$ $\exists$ a contraction K s. th.

$$A + K = (A_{22} - \mu)^{-1} (1 - KG)^{-1} K (S - \mu).$$

<u>Lemma</u> 6. Denote $H_S = \mathcal{D}(\bar{S}) \subset H_+$, where $\bar{S}$ is the closure of S and the norm in H_S is defined by

$$\|x_+\|_{H_S} = \sqrt{\|\bar{S} x_+\|^2 + \|x_+\|^2}.$$

Then $\exists$ a complete orthogonal system $\{\varphi_k\}_1^\infty$ in H_+ such that $\{\varphi_k\}_1^\infty$ is a Riesz basis in H_S.

<u>Proof</u>. If H_S is compactly embedded in H_+ we take $\{\varphi_k\}_1^\infty$ consisting of eigenvectors of $S^* \bar{S}$. In general case

additional work is required.

Proof of Theorem A.

Let P_n be orthogonal projectors onto $\mathcal{L}in\{\varphi_k\}_1^n$ in H_+. Then $P_n \to 1$ in H_+ and $P_n \to 1$ in H_S.

Consider

$$A_n = \begin{pmatrix} P_n A_{11} P_n & P_n A_{12} \\ A_{21} P_n & A_{22} \end{pmatrix} \quad \text{in } H_n^+ \oplus H^-,$$

$H_n^+ = P_n(H^+)$.

Then A_n is m-dissipative in Pontrjagin space Π_n and due to Krein-Langer-Azizov theorem (a) holds.

This implies (Lemma 3) that

$$(*)\ F_n + K_n = (A_{22}-\mu)^{-1}(1-K_n G)^{-1} K_n (S_n-\mu).$$

Choose $K_n \rightharpoonup K$. Since $\|K_n\| \leq 1$, we have $\|K\| \leq 1$. Then

$$F_n = F P_n \to 1$$

$K_n G \rightrightarrows KG$ and $(1-K_n G)^{-1} \rightrightarrows (1-KG)$

(we essentially use here that G is compact!)

Further,

$$K_n S_n = K_n S P_n ,$$

$$\bar{S} P_n x \to \bar{S} x \quad \forall x \in \mathcal{D}(\bar{S}),$$

Hence, $K_n S P_n x \rightharpoonup K S x$.

Therefore we can pass to the weak limit in the equation (*) and obtain

$$F + K = (A_{22} - \mu)^{-1} (1 - KG)^{-1} K (S - \mu)$$

and by virtue of Lemma 3 property (a) holds. $\square$

Let $A^+ = \bar{A}/_{\mathcal{L}^+}$. How to prove

(b): $\exists\ \mathcal{L}^+$ such that $\sigma(A^+) \subset \bar{\mathbb{C}}^+$?

We have

$$(\bar{A} - \mu) \begin{pmatrix} x_+ \\ K x_+ \end{pmatrix} = \begin{pmatrix} (\bar{S} - \mu + GL) x_+ \\ L x_+ \end{pmatrix},$$

where $L := (A_{22} - \mu)(F + K)$, $\mathcal{D}(L) = \mathcal{D}(\bar{S})$.

Consider

$Q : \mathcal{L} \to H_+$ defined by $Q \begin{pmatrix} x_+ \\ K x_+ \end{pmatrix} = x_+$.

Q is bounded and boundedly invertible, $\| Q^{-1} \| \le 2$.

We have

$$\bar{A}/_{\mathcal{L}^+} = Q^{-1}(S+GL)\,Q =$$
$$= Q^{-1}[1+G(1-KG)^{-1}K(\bar{S}-\mu)]\,Q,$$

hence

$$(**)\ (\bar{A}-\alpha)/_{\mathcal{L}^+} = Q^{-1}[1+T(\alpha)]\,(\bar{S}(\mu)-\alpha)\,Q,$$

where

$$T(\alpha) = G(1-KG)^{-1}K(\bar{S}-\mu)(\bar{S}-\alpha)^{-1}$$

is a holomorphic operator function whose values are compact operators. Here we assumed that $(\bar{S}-\alpha)^{-1}$ exists $\Leftrightarrow$ $\bar{S}$ is m-dissipative in H_+ $\Leftrightarrow$ $\bar{A}$ is m-dissipative in $\mathcal{K}$. It can be shown that $\|T(\alpha)\| \to 0$ as $\alpha \to \infty$ along negative imaginary exis, therefore $1+T(\alpha)$ has only discrete spectrum in $\mathbb{C}^-$. We use the following

<u>Lemma</u> 7. $\operatorname{Im}[Ax_0, x_0] = \operatorname{Im}\alpha_0\,[x_0, x_0]$ if $Ax_0 = \alpha_0 x_0$.

Therefore, all eigenvectors of A corresponding to the eigenvalues from $\mathbb{C}^-$ are of negative type provided that A is strictly dissipative in $\mathcal{K}$.

This proves Theorem B if we assume in addition that A is strictly dissipative in $\mathcal{K}$.

If not, we consider

$$A_\varepsilon = A + i\varepsilon P_+, \quad \varepsilon > 0.$$

Assertion (a) is valid for A_ε and it does not have spectrum in $\mathbb{C}^-$, since

$$\operatorname{Im}\left[(A + i\varepsilon P_+)\begin{pmatrix} x_+ \\ Kx_+ \end{pmatrix}, \begin{pmatrix} x_+ \\ Kx_+ \end{pmatrix}\right] \geqslant \varepsilon (x_+, x_+).$$

Write Riccati equation for A_ε:

$$F + K_\varepsilon = (A_{22} - \mu)^{-1}(1 - K_\varepsilon G)^{-1} K_\varepsilon (S + i\varepsilon - \mu).$$

Take $\varepsilon_n \to 0$ and $K_{\varepsilon_n} =: K_n \to K$.

We have

$$A_\varepsilon^+ = Q^{-1}[1 + T_\varepsilon(\alpha)](S + i\varepsilon - \alpha)Q$$

and

$$T_\varepsilon(\alpha) = G(1 - K_\varepsilon G)^{-1} K_\varepsilon (S + i\varepsilon - \mu)(S + i\varepsilon - \alpha)^{-1}$$

$$\Longrightarrow T(\alpha).$$

Since $1 + T_\varepsilon(\alpha)$ is a holomorphic operator function of Fredholm type in $\mathbb{C}_-$, boundedly invertible $\forall\, \alpha \in \mathbb{C}^-$, so is $1 + T(\alpha)$. $\square$

Theorems C–E are proved by analyzing representation (**).