Spectral problems for <u>high</u> <u>contrast</u> periodic media and <u>HOMOGENISATION</u>

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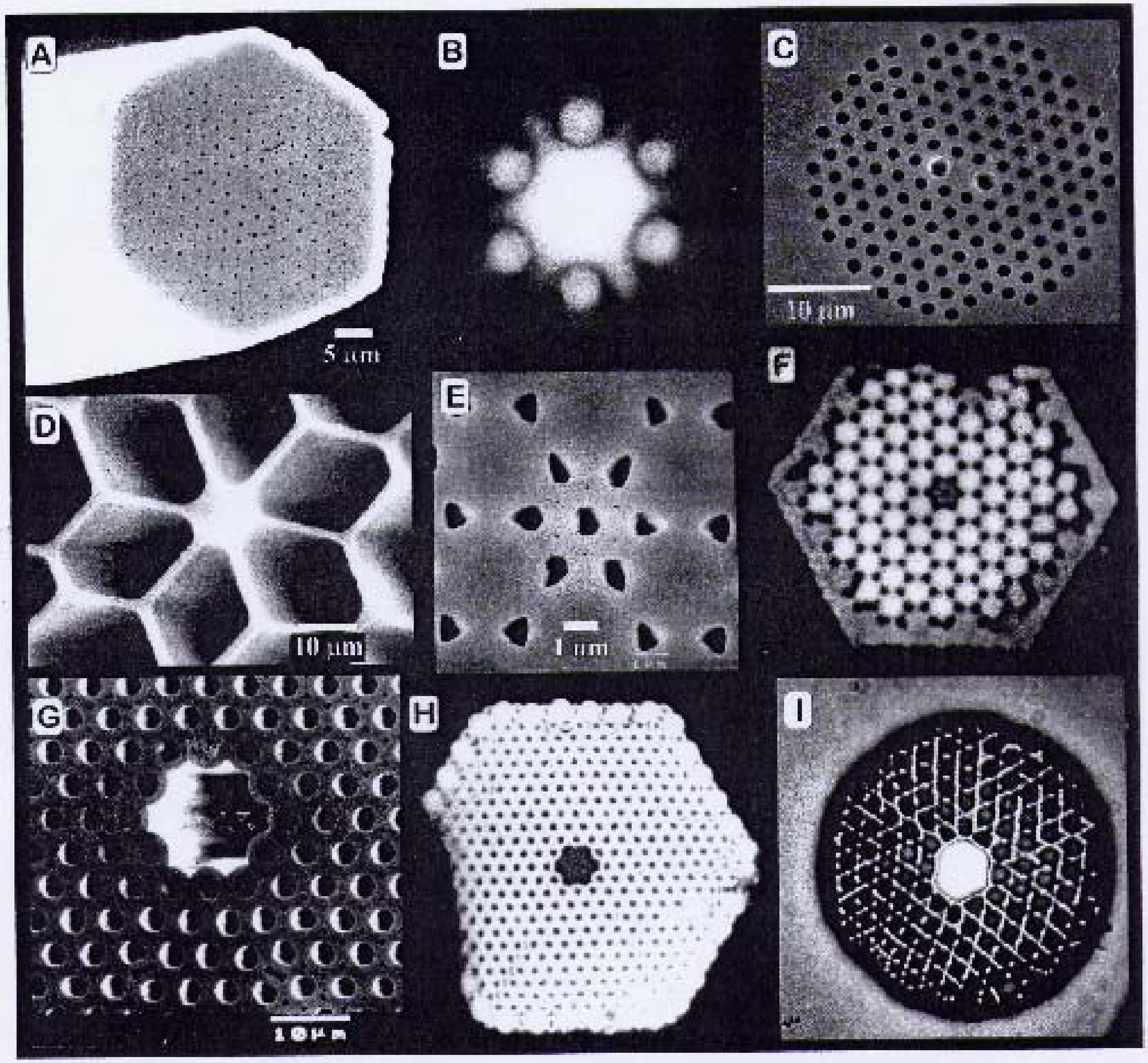
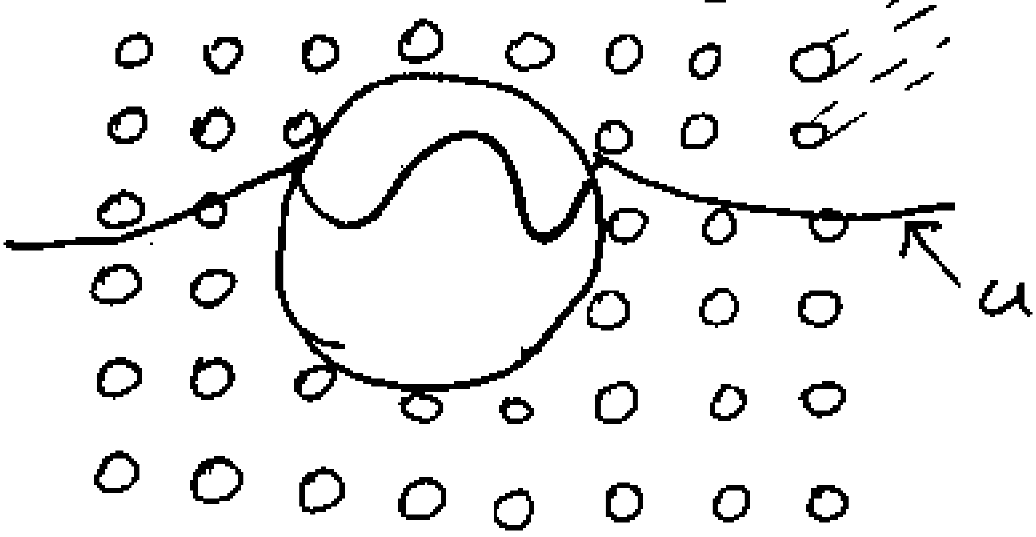
Photonic crystals
(Photonic crystal fibres)



(from P. Russell,
"Photonic crystal fibers",
Science, 299, 358-362, 2003)



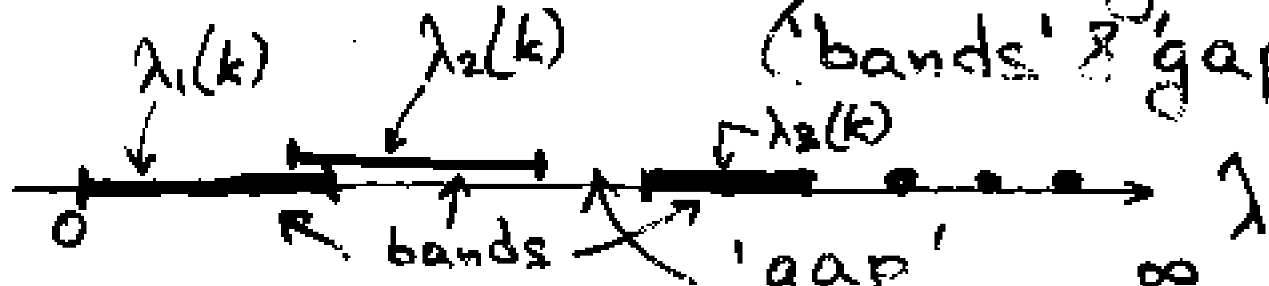
$\Rightarrow$ Typical (model) eigenvalue problem on the 'cross-section':

$$-\mathrm{div}\big((a(x) + b(x))\nabla u\big) = \lambda u$$

periodic — "defect" (localized)

from the <u>SPECTRAL THEORY</u>:

1. $b \equiv 0$ (periodic) $\Rightarrow$ Bloch-Floque theory ('bands' & 'gaps')



$$L_0 u := -\mathrm{div}(a\nabla u) \Longrightarrow Sp(L_0) = \bigcup_{j=1}^{\infty} \bigcup_{k\in Q^*} \lambda_j(k$$

$$L_0 u = \lambda_j(k) u, \quad u(x,k) = e^{ik\cdot x}\, \underbrace{v(x,k)}_{\text{periodic}}$$

(see Kuchment "Math of Photonic crystals", 2001)

M. S. Birman '61 – general approach

Alama, Avellaneda et al '94 – scalar (acoustic)

Figotin & Klein '97 – vector (Maxwell)

$$L_0 u := -\nabla\cdot(a(x)\nabla u)$$

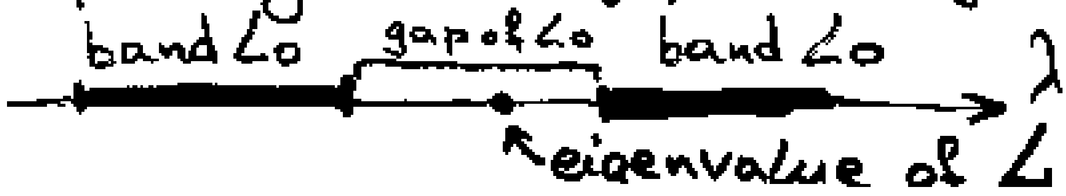
$$L u := -\nabla\cdot((a(x)+b(x))\nabla u)$$

$$0<\nu \le a(x),\ a(x)+b(x) \le \nu^{-1}$$

$b(x)$ – compactly supported

$\Rightarrow$ (i) $\sigma_{ess}(L) = \sigma_{ess}(L_0)$

Coroll: There may exist only isolated eigenvalues of finite multiplicity for L, in the gaps of L_0

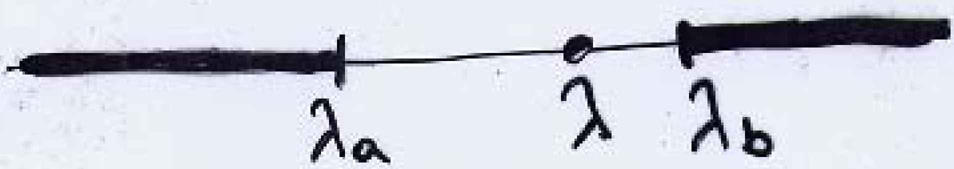


ii) eigenvalues ...
"sometimes"

iii) When the eigenvalues in the gap do exist, the eigenfunctions decay exponentially:

$$\mathcal{L}u = \lambda u$$

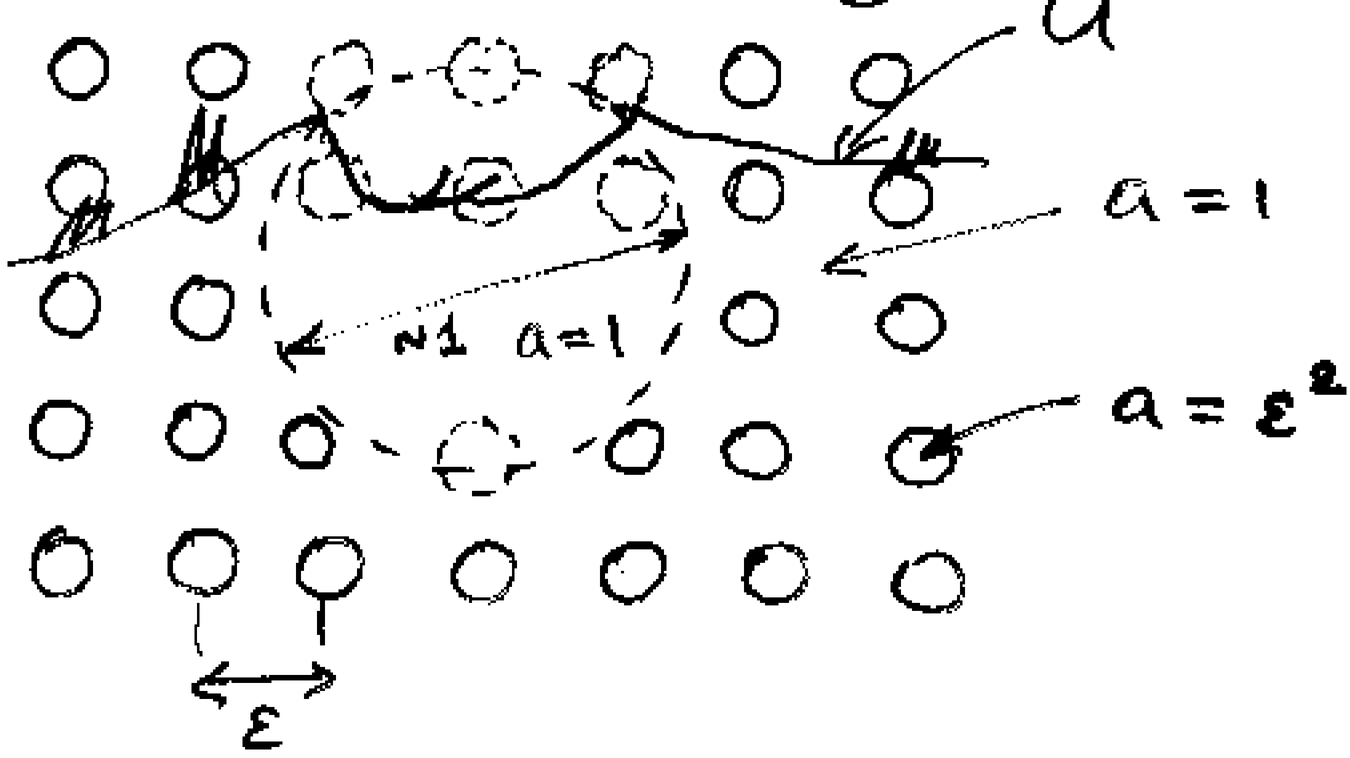
$$\Rightarrow \quad u(x) = O\left(e^{-C\left(|\lambda-\lambda_a||\lambda-\lambda_b|\right)^{1/2}|x|}\right), \quad |x| \to \infty$$



(Barbaroux, Combes, Hislop '97; Combes, Hislop, Tip '99)

Q: What can 'HOMOGENISATION' ADD TO all this?

A. — (Existence + asymptotics of eig values/f-ns;



$$L^\varepsilon = -\nabla \cdot a^\varepsilon(x) \nabla$$

('double-porosity' type scaling)

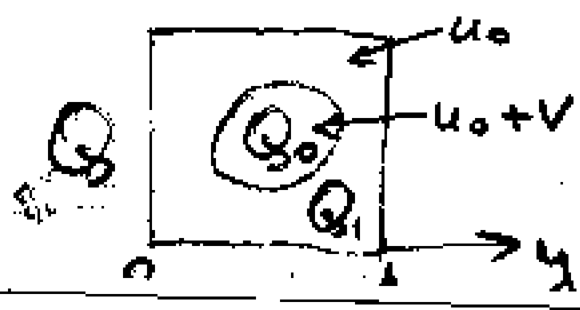
[<u>Formal calculation</u>:

$$L^\varepsilon u^\varepsilon = \lambda u^\varepsilon$$

$$u^\varepsilon(x) \sim u^{(0)}\left(x, \tfrac{x}{\varepsilon}\right) + \varepsilon u^{(1)}\left(x, \tfrac{x}{\varepsilon}\right) + \dots$$

$u^{(0)}(x,y), u^{(1)}(x,y), \dots$ y-periodic(Q)

$$\Rightarrow \dots \quad u^{(0)}(x,y) = \begin{cases} u_0(x), & y \in Q_1 \text{ or } |x| < \\ u_0(x) + V(x,y), & y \in Q_0, \ |x| > 1 \end{cases}$$



where (u_0, V) solve

↓

coupling

$$(*)\begin{cases} -\nabla\cdot(A^*\nabla u_0) = \lambda\left(u_0+\langle v\rangle_y\right), & |x|>R \\ -\Delta_y v = \lambda(u_0+v), & y\in Q_0,\ |x|>R \\ [u_0]=0,\quad \left(\dfrac{\partial u_0}{\partial n}\right)_- = \left(A^*_{ij}\dfrac{\partial u_0}{\partial x_j}n_i\right)_+, & |x|=R \end{cases}$$

$$\langle v\rangle_y(x) := \frac{1}{|Q|}\int_{Q_0} v(x,y)\,dy$$

$A^*=(A^*_{ij})$ – "porous" homogenised matrix:

$(\xi\in\mathbb{R}^n)$

$$A^*\xi\cdot\xi = \inf_{w\in C^\infty_{per}(Q)}\int_{Q_1}|\xi+\nabla w|^2\,dy.$$

$\cancel{\mathcal{L}}$ (*) — 'Limit' spectral problem:

$$\mathcal{L}u^{(0)} = \lambda u^{(0)},$$

$u^{(0)} = u^{(0)}(x,y)$

$$\mathcal{L} : D(\mathcal{L})\subset H\subset L^2(\mathbb{R}^n, L^2(Q)) \longrightarrow H \text{ – limit operator}$$

$$H = \Big\{ u(x,y) \in L^2(\mathbb{R}^n \times Q) \;\Big|\; u = u_0(x) + v(x,y)$$
$$u_0 \in L^2(\mathbb{R}^n)$$
$$v \in L^2(\mathbb{R}^d \setminus B_R, L^2(Q_0)) \Big\}$$

Quadratic form:

$$V = H^1(\mathbb{R}^n) + L^2(\mathbb{R}^d \setminus B_R, H_0^1(Q_0))$$

$$u \in V \longmapsto u = (u_0, v)$$

$$Q(u,u) = \int_{B_R} |\nabla u_0|^2 dx + \int_{\mathbb{R}^n \setminus B_R} A^* \nabla u_0 \cdot \nabla u_0$$
$$+ \int_{Q_0} \int_{\mathbb{R}^n \setminus B_R} |\nabla_y v|^2 dx\, dy$$

(Q-closed $\longmapsto$ Self-adjoint on $\mathcal{D} \subset V$, etc..)

(formally) eliminating $v(x,y)$ from (*):

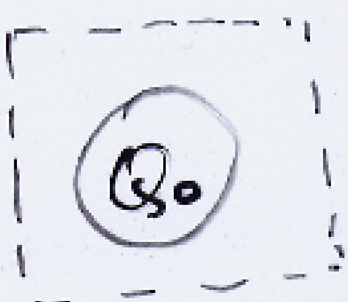
$$v(x,y) = \lambda u_0(x) \sum_{j=1}^{\infty} \frac{\langle \varphi_j \rangle_y}{\lambda_j - \lambda} \varphi_j(y)$$

$$\Rightarrow \begin{cases} -\Delta u_0 = \lambda u_0, & |x| < R \\ -\nabla \cdot (A^* \nabla u_0) = \beta(\lambda) u_0, & |x| > R \\ [u_0] = 0, \ \left(\frac{\partial u}{\partial n}\right)_- = \left(n \cdot A^* \nabla u_0\right)_+, & |x| = R \end{cases}$$

where

$$\beta(\lambda) := \lambda + \lambda^2 \sum_{j=1}^{\infty} \frac{\langle \varphi_j \rangle_y^2}{\lambda_j - \lambda}$$

(Zhikov '00, 04; no 'defects')

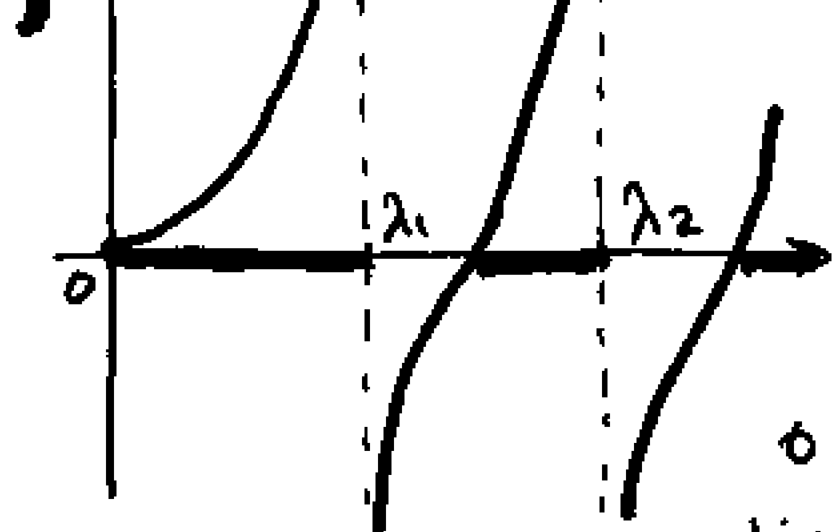


$$-\Delta_y \varphi_j = \lambda_j \varphi_j, \ Q_0$$

$$\varphi_j = 0, \ \partial Q_0$$

$$0 < \lambda_1 \leq \lambda_2 \leq \dots$$

$\{\varphi_j(y)\}$ — o.n. basis in $L^2(Q_0)$



$$\begin{cases} -\Delta\varphi_j = \lambda_j \varphi_j \\ \varphi_j = 0, \ \partial Q_0 \end{cases}$$

$0 < \lambda_1 \leq \lambda_2 \leq \dots$

$\{\varphi_j\}$ - orthonormal in $L^2(Q_0)$;

~~end of formal calculation~~

$\beta(\lambda) < 0$

$$\Rightarrow \quad u_0(x) \sim e^{-|\beta(\lambda)|^{1/2}|x|} \, |x|^{\frac{1-n}{2}}; \quad |x| \to \infty$$

Seek in the 'gap' $(\beta(\lambda) < 0)$ the (formal) eigenvalues:

$$\begin{cases} (\Delta + \lambda_0) u_0 = 0, & |x| < 1 \\ (\nabla \cdot A^* \nabla + \beta(\lambda_0)) u_0 = 0, & |x| > 1 \\ [u] = \left[A \frac{\partial u}{\partial n}\right] = 0, & |x| = 1 \end{cases}$$

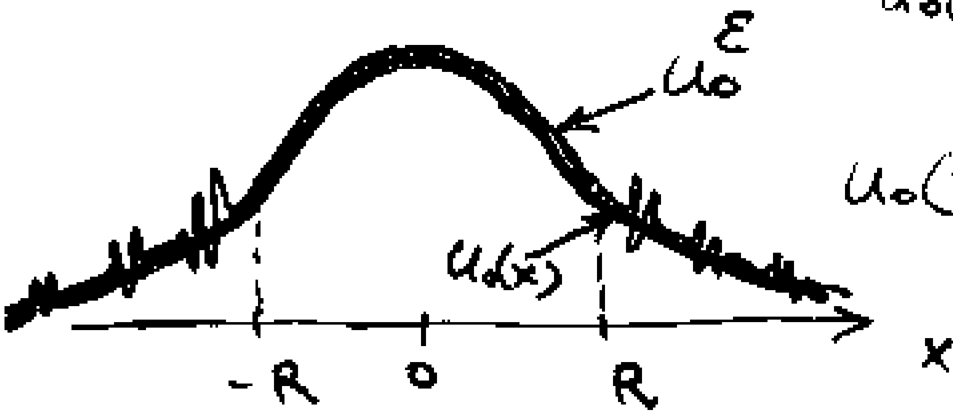
(Exist e.g. Bessel / modified Bessel $(n=2)$

$$u_0 = A \frac{e^{-|\beta|^{1/2}|x|}}{|x|} H(|x| - R) + \frac{\sin(\lambda^{1/2}|x|)}{|x|} H(R - |x|)$$

$(n=3)$

$$u_0^\varepsilon(x) = \begin{cases} u_0(x), & |x| \le 1+ \\ u_0(x) + v\left(x, \frac{x}{\varepsilon}\right), & |x|>1+2\varepsilon \end{cases}$$

$$\text{“}u_0\left(x, \tfrac{x}{\varepsilon}\right)\text{)}$$



$$u_0(x) \sim e^{-|\beta(\lambda_0)|^{1/2}|x|}$$
$$|x| \to \infty$$

end of formal derivation.]

Justification:

<u>Theorem</u> (V.S. & I. Kamotski, 2005)

For any $(\lambda_0, u_0(x,y))$ "limit two-scale problem" eigenvalues there exists $\varepsilon_0 > 0, \delta > 0$ such that $\forall\ 0 < \varepsilon < \varepsilon_0$

(a) $\lambda_0 \notin \sigma_{ess}(\mathcal{L}^\varepsilon)$

$$[\lambda_0 - \delta, \lambda_0 + \delta] \cap \sigma_{ess}(\mathcal{L}^\varepsilon) = \emptyset$$

(b) There exists $\lambda_\varepsilon \in \sigma_p(\mathcal{L}^\varepsilon)$ s.t.

$$|\lambda_\varepsilon - \lambda_0| < C\varepsilon^{1/2}$$

Idea of proof

(a) 1) $\mathcal{L}_0^\varepsilon := -\nabla \cdot a_0\left(\frac{x}{\varepsilon}\right)\nabla$

periodic (no defect)

$$\Rightarrow \sigma(\mathcal{L}_0^\varepsilon) \xrightarrow{\text{Hausdorf}} \{\lambda : \beta(\lambda) \geq 0\} \cup \{\lambda_j\}$$

(Hempel & Lienau '00; Zhikov '00, 04)

2) $\sigma_{ess}(\mathcal{L}^\varepsilon) = \sigma_{ess}(\mathcal{L}_0^\varepsilon)$

$\square$

(b) Consider

$$I(\varepsilon) := \inf_{\substack{u \in D(\mathcal{L}^\varepsilon) \\ \|u\|_{L^2(\mathbb{R}^n)} = 1}} \|(\mathcal{L}^\varepsilon - \lambda_0)u\|_{L^2(\mathbb{R}^n)}$$

• $\begin{pmatrix}\text{Raileigh-} \\ \text{-type}\end{pmatrix}$ variational principle:

$$I(\varepsilon) = \text{dist}(\lambda_0, \sigma(\mathcal{L}^\varepsilon)) =: d^\varepsilon$$

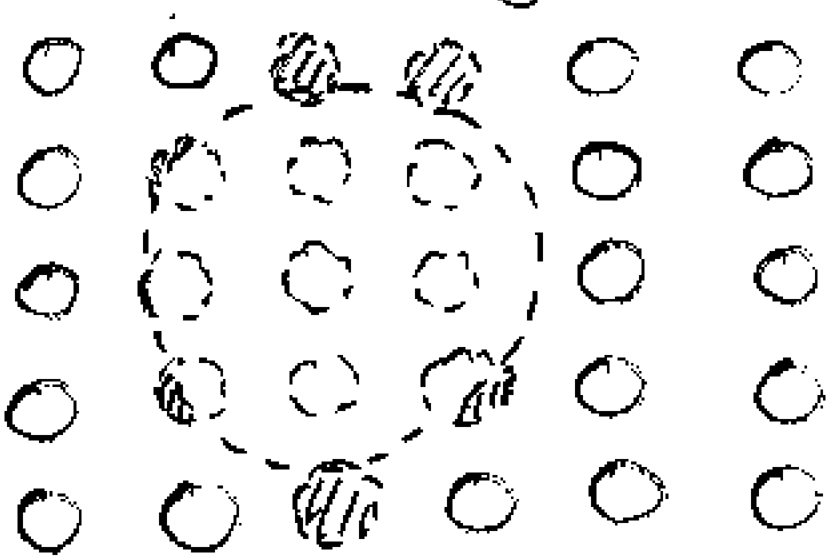
$\Rightarrow$ • For a 'test function' $\tilde{u}^\varepsilon \in D(\mathcal{L}^\varepsilon)$

$$d^\varepsilon \leq \frac{\|(\mathcal{L}^\varepsilon - \lambda_0)\tilde{u}^\varepsilon\|}{\|\tilde{u}^\varepsilon\|} =: \tilde{d}_0^\varepsilon$$

- Select $u(x)$ via formal asymptotics:

$$\tilde{u}^{\varepsilon}(x) := \overbrace{u^{(0)}\left(x, \tfrac{x}{\varepsilon}\right)}^{=u_0^{\varepsilon}} + \varepsilon u^{(1)}\left(x, \tfrac{x}{\varepsilon}\right) + \varepsilon^{2} u^{(2)}\left(x, \tfrac{x}{\varepsilon}\right) + \text{"boundary layer"}$$

- By construction (two-scale formal asymptotics)



$$d_0^{\varepsilon} \leq \underbrace{C_1 \varepsilon^{1/2}}_{\text{from boundary layer}} + \underbrace{C_1 \varepsilon}_{\text{from "bulk" } (|x| > 1 + O(\varepsilon}$$

$\Rightarrow \exists \lambda_{\varepsilon} \in \sigma(\mathcal{L}^{\varepsilon})$ s.t.

$|\lambda_{\varepsilon} - \lambda_0| \leq C \varepsilon^{1/2}$

$\lambda_{\varepsilon} \in \sigma_p(\mathcal{L}^{\varepsilon})$ by (a) $\blacksquare$

What about convergence of eigenfunctions and their 'asymptotic completeness'

above

Asymptotic methods apparently no use...

Instead:

'Strong two-scale resolvent convergence' (Zhikov '00, 04 no defects!)

$$L^\varepsilon : \mathcal{D}(L^\varepsilon) \subset L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$$

'Limit' operator:

$$\mathcal{L} : \mathcal{D}(\mathcal{L}) \subset H_0 \subset L^2(\mathbb{R}^n \times Q) \to H_0$$

Two-scale convergence

$$u^\varepsilon(x) \sim u^{(0)}\left(x, \tfrac{x}{\varepsilon}\right), \quad u^{(0)}(x,y) \in L^2(\mathbb{R}^n \times Q)$$

periodic

Def 1: $H \ni \underset{\text{bdd}}{u^\varepsilon(x)} \xrightarrow{2} u^{(0)}(x,y)$

$\forall \varphi \in C_0^\infty(\mathbb{R}^n),\ b \in C^\infty_{per}(Q)$

$$\int_{\mathbb{R}^n} u^\varepsilon(x)\varphi(x)\, b\left(\tfrac{x}{\varepsilon}\right)dx \to \iint_{\mathbb{R}^n \times Q} u^{(0)}(x,y)\varphi(x)b(y)\,dx\,dy$$

Def 2: $\underset{\text{bdd}}{u^{\varepsilon}} \xrightarrow{2} u^{(0)}(x,y)$

$\forall\, v_\varepsilon(x) \xrightarrow{2} v(x,y)$

$$\int u^\varepsilon(x) v^\varepsilon(x)\,dx \to \int_{\mathbb{R}^n} \int_Q u(x,y) v(x,y)\,dx\,dy$$

$$\left(\|u^\varepsilon(x) - u(x, \tfrac{x}{\varepsilon})\|_{L^2} \to 0 \right)$$

Def 3: (strong 2-scale resolvent convergence)

$L^\varepsilon, \mathcal{L} \geq 0$

$\forall\, f_\varepsilon \xrightarrow{2} f$

$$(*) \quad (L^\varepsilon + I)^{-1} f_\varepsilon \xrightarrow{2} (\mathcal{L} + I)^{-1} f$$

Th-m (Zhikov '00, '04, no defects)

$$L^\varepsilon \xrightarrow{S2SRC} \mathcal{L}$$

- (*) implies convergence of spectral projectors:

$$\lambda \notin \sigma_p(\mathcal{L}) \Longrightarrow P_\varepsilon(\lambda) f_\varepsilon \xrightarrow{2} P(\lambda) f$$

$$\forall\, f_\varepsilon \xrightarrow{2} f$$

S2SRC not enough for
onvergence & "asympt. completeness"
of eigenvalues!

+

<u>Th-m</u> (Strong 2-scale compactness
of eigenf-ns; Zhikov, no defects):

$\lambda_\varepsilon \in \sigma_d(L_\varepsilon)$

$(\lambda_\varepsilon, u^\varepsilon)$, $\|u^\varepsilon\|_{L^2} = 1$, λ_ε - bdd

$\Rightarrow$ $\{u^\varepsilon\}$ is <u>compact</u> in the
sense of 2-sc. conv, i.e.

$\exists\ \lambda_{\varepsilon_j} \longrightarrow \lambda$

$u_{\varepsilon_j} \xrightarrow{2} u^{(0)}(x,y)$

(hence $(\lambda, u^{(0)}) \in \sigma_d(\mathcal{L}_0)$)

$\|u^{(0)}(x, x/\varepsilon_j)\| \longrightarrow 1$.

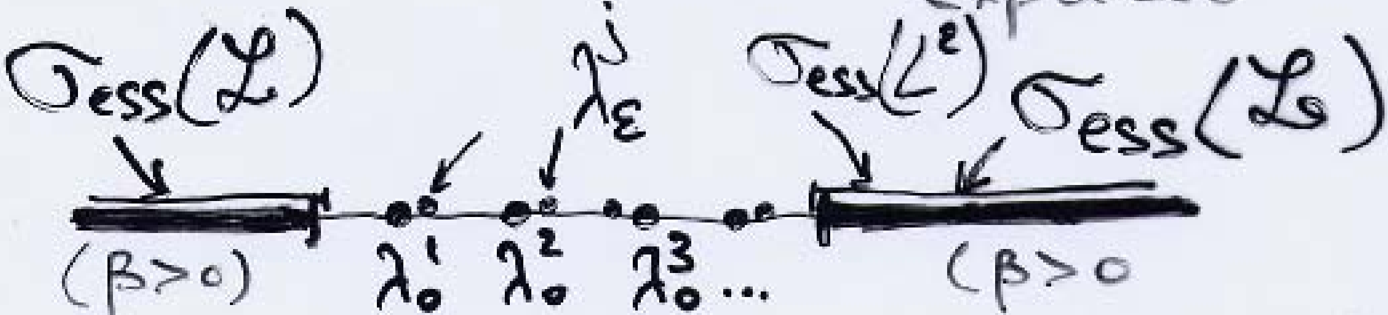
$\Downarrow$

Th-m 2:

1. $\|u^\varepsilon(x) - u_0^\varepsilon(x)\|_{L^2(\mathbb{R}^n)} \le C\varepsilon^{1/2}$

(convergence of eigenvalues)

2. $\sigma(L^\varepsilon) \xrightarrow{\text{Hausdorf}} \underbrace{\sigma(\mathcal{L})}_{\text{"explicit"}}$



('asymptotic completeness)