The transfer operator approach to Q-chaos on modular surfaces

Geodesic flow

$\Phi_t : SM \to SM$

$\Big\downarrow$ Thermodyn. formalism

$\boxed{\text{Transfer operator } \mathcal{L}_\beta \quad \det(1-\mathcal{L}_\beta) = Z(\beta)}$

Selberg - Gutzwiller

$Z(\beta) = 0$

$\mathcal{L}_\beta f = f \longrightarrow$

Laplace - Beltrami operator

$\boxed{\Delta_{LB}\, u = \beta(1-\beta)\, u}$

$\Big\updownarrow$ harmonic analysis

$\boxed{\text{Lewis's funct. equation} \quad \varphi(z) = \varphi(z+1) + (z+1)^{-2\beta}\varphi\left(\frac{z}{z+1}\right)}$

$$Z(\beta) = \dot{\prod_{\gamma}} \prod_{k=0}^{\infty} \left(1 - e^{-(\beta+k)\, l(\gamma)}\right)$$

$l(\gamma)$ = period of γ

1. Transfer operator for geod. flow on mod. surface

$\mathbb{H} = \{ z = x+iy, y>0 \}$ hyperbolic plane

$ds^2 = \frac{dx^2+dy^2}{y^2}$ hyperbolic metric

$PSL(2,\mathbb{R})$ group of isometries

$g \overset{\Psi}{=} \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}$: $gz = \frac{az+b}{cz+d}$ Möbius transf.

$PSL(2,\mathbb{Z})$ full modular group $(a,b,c,d \in \mathbb{Z})$

$\Gamma \subset PSL(2,\mathbb{Z})$ finite index μ_Γ.

Symbolic dynamics (Artin, ...)

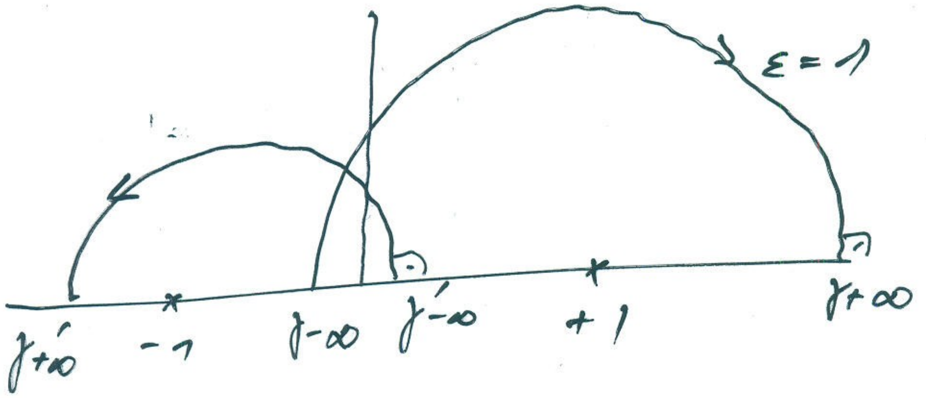
$P_\Gamma : \Sigma_\Gamma \to \Sigma_\Gamma$ Poincaré map

$$\Sigma_\Gamma = I_2 \times \mathbb{Z}_2 \times \Gamma \backslash PSL(2,\mathbb{Z})$$

$$P_\Gamma(x,y,\varepsilon,[g]) = \left(T_G(x), \frac{1}{y+[\frac{1}{x}]}, -\varepsilon, [g\, T^{-n\varepsilon} S]\right)$$

$\gamma_{-\infty} = -\varepsilon y$, $\gamma_{+\infty} = \varepsilon \frac{1}{x}$; $T_G x = \frac{1}{x} \bmod 1$

$\gamma = (\gamma_{-\infty}, \gamma_{+\infty})$ endpoints of geodesic γ



reduced geodesics

$|\gamma_{-\infty}| < 1 \qquad |\gamma_{+\infty}| > 1$

<u>transfer</u> <u>operator</u> $\mathcal{L}_\beta$

$$(\mathcal{L}^\Gamma_\beta \underline{f})(x,\varepsilon) = \sum_{n=1}^{\infty} \left(\frac{1}{x+n}\right)^{2\beta} \chi_\Gamma(S T^{n\varepsilon})\, \underline{f}\left(\frac{1}{x+n}, -\varepsilon\right)$$

$$\underline{f}(x,\varepsilon) = \left(f(x,\varepsilon,[g])\right)_{[g]\in \Gamma \backslash PSL(2,\mathbb{Z})}$$

$\mathrm{Re}\,\beta > \frac{1}{2}$, χ_Γ repr. induced from ~~triv.~~ rep. of Γ

- $\mathcal{L}_\beta^\Gamma$ meromorphic family of trace class op. in B-space of holom. fcts.
- $Z(\beta) = \det(1-\mathcal{L}_\beta^\Gamma)$ Selberg zeta of Γ

Manin-Marcolli: $SL(2,\mathbb{Z}) \to GL(2,\mathbb{Z})$

$$\Gamma \to \tilde{\Gamma} = \Gamma \cup \Gamma\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \longrightarrow \bar{\Gamma} = \tilde{\Gamma} \cup \tilde{\Gamma}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathcal{L}_\beta^\Gamma \underline{f}(z) = \sum_{n=1}^{\infty} \left(\frac{1}{z+n}\right)^{2\beta} \chi_{\bar{\Gamma}}\left(\begin{pmatrix} -n & 1 \\ 1 & 0 \end{pmatrix}\right) \underline{f}\left(\frac{1}{z+n}\right)$$

$$Z(\beta) = \det(1-\mathcal{L}_\beta^\Gamma)\det(1+\mathcal{L}_\beta^\Gamma)$$

$$Z(\beta) = 0 \longleftrightarrow 1 \text{ or } -1 \text{ in } \sigma(\mathcal{L}_\beta^\Gamma).$$

2. Eigenfcts. of $\mathcal{L}_\beta$ with EV $\lambda = \pm 1$

$\mathcal{L}_\beta^\Gamma \underline{f} = \pm 1 \underline{f} \iff \beta$ is zero of Selberg fct. $Z(\beta)$

$\mathrm{Re}\,\beta < 0$ trivial zero

$\mathrm{Re}\,\beta > 0$ spectral zero

For $\underline{f}$ such an eigenfct. put $\varphi(z) = \underline{f}(z-1)$

$\Rightarrow \varphi$ fulfills a fctional equation

$$\varphi(z) - \chi_{\bar\Gamma}(T^{-1})\,\varphi(z+1) - \lambda z^{-2\beta}\,\chi_{\bar\Gamma}(T^{-1}M)\,\varphi(1+\tfrac{1}{z}) = \underline{0}$$

where $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

slash action of $GL(2,\mathbb{Z})$ on components of φ

$$(\varphi|_\beta h)(z, \bar\Gamma g) := (cz+d)^{-2\beta}\,\varphi(hz, \bar\Gamma g T^{-1} h T)$$

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{Z})$$

Lewis functional equation for Γ

$$\boxed{\varphi - \varphi|_\beta T - \lambda\,\varphi|_\beta TM = 0}\,.$$

<u>Lemma</u>: $f(z) = \big(\varphi(z+1), \bar\Gamma g\big)_{\bar\Gamma g \in \bar\Gamma \backslash GL(2,\mathbb{Z})}$ is an eigen-fct. of $\mathcal{L}_\beta^\Gamma$ with EV $\lambda = \pm 1$ iff φ is a solution of the Lewis Eq. with certain growth properties, holomorphic in cut plane $\mathbb{C} \setminus [-\infty, 0]$.

3. <u>Maass wave forms and eigenfcts of $\mathcal{L}_\beta$</u>

$u : \mathbb{H} \to \mathbb{C}$ is <u>Maass wave form</u> with spectral parameter β if

- $\Delta_{LB}\, u = \beta(1-\beta)\, u \qquad \Delta_{LB} = -y^2(\partial_x^2 + \partial_y^2)$

- $u(gz) = u(z) \qquad \forall g \in \Gamma$
- u vanishes at the cusps of Γ

$S(\Gamma, \beta)$ space of Maass wave forms with spectral parameter β

$\varrho_\Gamma : PSL(2,\mathbb{Z}) \longrightarrow \mathrm{End}(\mathbb{C}^{\mu_\Gamma})$

repres. of $PSL(2,\mathbb{Z})$ induced from trivial repres. of Γ

$$\varrho_\Gamma(g) = \left(\delta_\Gamma(g_i g g_j^{-1}) \right)_{1 \le i,j \le \mu_\Gamma} .$$

map $\Pi : S(\Gamma,\beta) \to S_{ind}(\Gamma,\beta)$

$(\Pi u)_i(z) := u|_0 g_i(z) \quad , 1 \le i \le \mu_\Gamma$

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$\varphi : \mathbb{C} \setminus (-\infty,0] \to \mathbb{C}^{\mu_\Gamma}$ is a <u>vector</u>

<u>fct. for Γ with parameter β</u> if

lomorphic

$(T^{-1})\varphi|_\beta T - \rho_\Gamma(T'^{-1})\varphi|_\beta T' = \underline{0}$

$' = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

tain growth properties for $z \to 0$

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ce of vector valued period. fct.

rameter β.

η 1-form

$$\eta(u,v) = (v\partial_y u - u\partial_y v)\,dx + (u\partial_x v - v\partial_x u)\,dy$$

u, v smooth fcts. on $\mathbb{H}$

- $\Delta_{LB} R_\zeta^\beta = \beta(1-\beta) R_\zeta^\beta$
- $\eta(u,v)$ is closed if $\Delta_{LB} u = \lambda u$, $\Delta_{LB} v = \lambda v$ (Lewis-Zagier)

$\underline{u} \in S_{ind}(\Gamma,\beta)$, $\underline{u} = (u_i)_{1\le i\le \mu_\Gamma}$

$$\boxed{(P\underline{u})_i(\zeta) = \int_0^{i\infty} \eta(u_i, R_\zeta^\beta)}\ .$$

Prop. (Mühlenbruch, D.M., Hilgert, Deitmar)

- $P\underline{u}$ is a vector valued period fct
- $P: S_{ind}(\Gamma,\beta) \longrightarrow FE_\Gamma(\beta)$ is bijection
- $P \circ \Pi : S(\Gamma,\beta) \longrightarrow FE_\Gamma(\beta)$ is bijection

Corollary: There is a bijection between the space of eigenfcts. of $\mathcal{L}_\beta^\Gamma$ with EV $\lambda = \pm 1$ and the space of Maass wave forms $S(\Gamma,\beta)$ for all β's such that β is a spectral parameter ($\mathrm{Re}\,\beta = \frac{1}{2}$)

4. The Hecke operators for $\Gamma_0(n)$

$$\Gamma_0(n) = \left\{ g \in SL(2,\mathbb{Z}) : g = \begin{pmatrix} a & b \\ nc & d \end{pmatrix} \right\}$$

Hecke congruence subgroup

$$\Gamma_0(n) = \bigcup_j \Gamma_0(nm)\, R_j^{n,m} \quad , \quad B_m = \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}, m \in \mathbb{N}$$

$$H_{n,m}\, u = u\big|_0 B_m \sum_j R_j^{n,m}$$ Hecke operators for $\Gamma_0(n)$

$H_{n,m} : S(\Gamma_0(n), \beta) \longrightarrow S(\Gamma_0(n), \beta)$

give a realization of the Hecke algebra

$H_{n,m} \circ H_{n,m'} = H_{n,m'} \circ H_{n,m}$

commute with Δ_{LB} : infinite family of symmetries

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\begin{tikzcd}
V(\Gamma_0(n),\beta) \arrow[r, leftrightarrow, "\mathcal{H}_n"] \arrow[d, "T_{n,m}"] & FE(\Gamma_0(n),\beta) & S_{ind}(\Gamma_0(n),\beta) \arrow[l, "P_n"'] \arrow[r, leftrightarrow, "\Pi_n"] \arrow[d, "\tilde{H}_{n,m}"] & S(\Gamma_0(n),\beta) \arrow[d, "H_{n,m}"] \\
V(\Gamma_0(n),\beta) \arrow[r, leftrightarrow, "\mathcal{H}_n"] & FE(\Gamma_0(n),\beta) & S_{ind}(\Gamma_0(n),\beta) \arrow[l, "P_n"'] \arrow[r, leftrightarrow, "\Pi_n"] & S(\Gamma_0(n),\beta)
\end{tikzcd}
$$

$T_{n,m}$ give a realization of the Hecke algebra on the space of EF of $\mathcal{L}_\beta^{\Gamma_0(n)}$ with EV $\lambda = \pm 1$.

Question: What role play these operators for the classical system?

Hilgert

endence for submodular groups

c. Mathematica (2006)

er, H. Movasati

for $\Gamma_0(n)$ and Hecke operators for the

s for $PSL(2,\mathbb{Z})$

. Phil. Soc. (2005) 139, 81–116

yer, T. Muehlenbruch

the Hecke algebra on the space of

ns for $\Gamma_0(n)$

lles J. Reine u. Angew. Math. (2006)

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