Universality in wave propagation for large times

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mixing and universality

A dynamical system $(X, \phi^t, \mathrm{d}\mu)$ is mixing if for $f, \rho \in L^2(X)$ ($\int_X \rho \, \mathrm{d}\mu = 1$)

$$\lim_{t \to \infty} \int_X f \circ \phi^{-t} \rho \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\mu$$

Interpretation: f observable, ρ probability density, $\int f \circ \phi^{-t} \rho \, \mathrm{d}\mu$ expected value of f at time t: the system forgets where is comes from - "Universality"

Wave propagation is for short wavelength driven by classical dynamics - geometric optics, semiclassics, etc. .

Main question: If the classical dynamics is mixing, what are the consequences for wave propagation at long times?

exponential mixing

Theorem. [Dolgopyat 98, Liverani 05] *Assume X is contact and $\phi^t : X \to X$ is Anosov and volume preserving, then there exists a $\gamma > 0$ such that for all $f, \rho \in C^1(X)$, $\int \rho \, \mathrm{d}\mu = 1$,*

$$\int f \circ \phi^{-t} \rho \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu + O(\|f\|_{C^1} \|\rho\|_{C^1} \mathrm{e}^{-\gamma|t|})$$

Localise initial conditions: $\rho_\varepsilon(x) := \frac{1}{\varepsilon^{2d-1}} \rho_0\big(\frac{d(x,x_0)}{\varepsilon}\big)$, then $\|\rho_\varepsilon\|_{C^1} \sim 1/\varepsilon^{2d}$, so

$$\lim_{\varepsilon \to 0} \int f \circ \phi^{-t} \rho_\varepsilon \, \mathrm{d}\mu = \begin{cases} f\big(\phi^{-t}(x_0)\big) & \text{for} \quad t << \frac{1}{\lambda} \ln \frac{1}{\varepsilon} \\ \int f \, \mathrm{d}\mu & \text{for} \quad t >> \frac{1}{\gamma} \ln \frac{1}{\varepsilon} \end{cases}$$

The Schrödinger equation

(M, g) smooth compact Riemannian manifold ($\partial M = \emptyset$). We are interested in the solutions of

$$\mathrm{i}\hbar\partial_t\psi(t) = -\frac{\hbar^2}{2}\Delta_g\psi(t)$$

with initial conditions of the form

$$\psi(t=0) = \psi_0 = a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}$$

where $a, \varphi \in C^\infty(M)$ and φ real valued. Since the initial conditions are oscillatory this is an hyperbolic problem (in the PDE sense), so we expect propagation. Let

$$\mathcal{U}(t) = \exp(\mathrm{i}\frac{\hbar t}{2}\Delta_g) \quad , \qquad \text{then} \quad \psi(t) = \mathcal{U}(t)\psi_0 \ .$$

geodesic flow

The associated classical sytem is the geodesic flow ϕ^t on

$$X = S^*M = \{(x,\xi) \in T^*M\,;|\xi|_{g(x)} = 1\}$$

,

$$\phi^t : S^*M \to S^*M \ .$$

ϕ^t is the Hamiltonian flow generated by

$$H(x,\xi) = \frac{1}{2}|\xi|^2_{g(x)}$$

If g has strictly negative sectional curvatures, then the geodesic flow is Anosov.

We are interested in the behaviour of $\psi(t) = \mathcal{U}(t)\psi_0$ for $\hbar \to 0$ and $t \to \infty$.

Why?

- $\hbar \to 0$, geodesic flow governs propagation
- $t \to \infty$, dynamical properties of the system unfold, e.g., chaos versus integrability.

Main question:

How does the behaviour of $\psi(t)$ depends on the underlying geometry of (M, g)?

Close to similar question in spectral geometry, spectral asymptotics, quantum chaos, etc.. .

Movies (by Arnd Bäcker)

Cardioid Billiard: in polar coordinates $r(\varphi) = 1 - \cos\varphi$, the billiard flow is mixing.

Δ-Laplace operator with Dirichlet boundary conditions.

$\psi_0(x) = \mathrm{e}^{-\alpha(x-x_0)^2/2}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}kx}$, $\alpha > 0$, $k \in \mathbb{R}^2$.

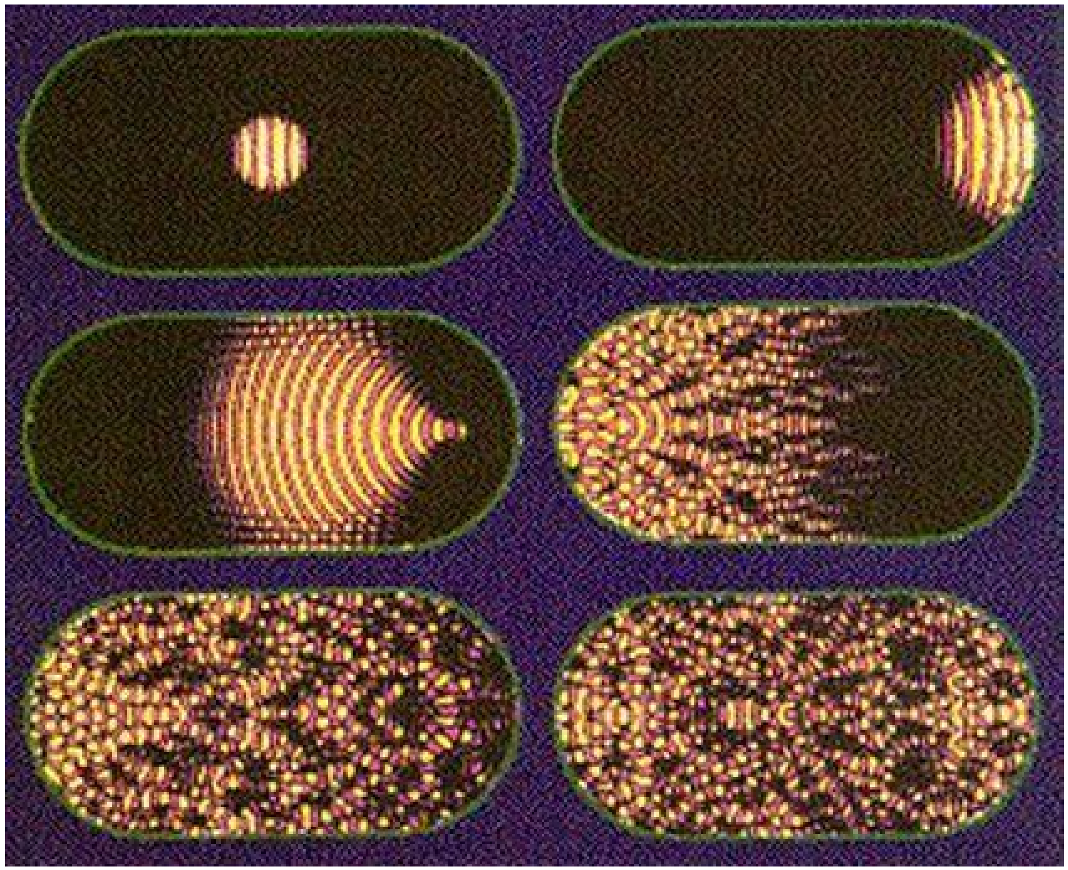
We plot $|\mathcal{U}(t)\psi_0|^2$, probability distribution in position space.

For observable $f(x)$, expected value at time t

$$\langle \mathcal{U}(t)\psi_0, f\,\mathcal{U}(t)\psi_0 \rangle = \int_M f(x)|\mathcal{U}(t)\psi_0|^2(x)\,\mathrm{d}x\ .$$

Equidistribution?

Tomsovic Heller 1991



Some history

- 1978-1979 Berman Zaslavsky, Balasy Berry Tabor Voros: log breaking time, Ehrenfest time $T_E \sim \frac{1}{\lambda} \ln \frac{1}{\hbar}$ limit of validity of semiclassics?
- 1979 Balasz Berry - random wave conjecture: equidistribution and universal fluctuations.
- 1991 Tomsovic Heller: validity of semiclassics beyond Ehrenfest time.
- 2000 Bonechi DeBiévre: equidistribution of coherent states in cat map on Ehrenfest time scales.

Lagrangian states

Let $\varphi \in C^\infty(M, \mathbb{R})$ then

$$\Lambda_\varphi := \{(x, d\varphi(x)) \ , x \in M\} \subset T^*M$$

is a **Lagrangian submanifold** (i.e. $\dim \Lambda = d$ and $\omega|_\Lambda = 0$ where $\omega = \sum_1^d \mathrm{d}x_i \wedge \mathrm{d}\xi_i$).

A **Lagrangian state** in $I^0(M, \Lambda)$ is a function of the form

$$\psi = a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}$$

with $a \in C^\infty(M)$ and where Λ is generated by φ.

Let $f(x,\xi) \in C_0^\infty(T^*M)$ and quantise it

$$\mathrm{Op}[f] = \text{"} f(x, \frac{\hbar}{\mathrm{i}}\nabla_x)\text{"} : L^2(M) \to L^2(M)$$

($\mathrm{Op}[f]$ is a pseudo-differential operator with principal symbol f).

Note $\frac{\hbar}{\mathrm{i}}\nabla_x\big(a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}\big) = (\nabla_x\varphi)a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi} + O(\hbar)$ and more generally one has

$$\mathrm{Op}[f]\big(a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}\big) = f(x, d\varphi(x))a(x)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi} + O(\hbar)$$

and

$$\langle\psi, \mathrm{Op}[f]\psi\rangle = \int_M f(x, d\varphi(x))|a(x)|^2 \,\mathrm{d}x + O(\hbar) =: \int_\Lambda f\rho(\psi) + O(\hbar)$$

which defines the the **density** $\rho(\psi)$ on Λ.

Theorem. [RS 05] *Assume ϕ^t is Anosov, $\Lambda \subset S^*M$, and Λ is transversal to the stable foliation (i.e. for all $x \in \Lambda$, $T_x\Lambda \cap E^s(x) = \{0\}$). Then there exists constants $\Gamma, \gamma > 0$ such that for all $f \in C_0^\infty(T^*M)$ and all $\psi \in I^0(M, \Lambda)$ with $\|\psi\|_{L^2} = 1$*

$$\langle \mathcal{U}(t)\psi, \mathrm{Op}[f]\mathcal{U}(t)\psi \rangle = \int_{S^*M} f \, \mathrm{d}\mu + O(\hbar \mathrm{e}^{\Gamma|t|}) + O(\mathrm{e}^{-\gamma|t|})$$

Remarks:

$$\int_M f(x)|\mathcal{U}(t)\psi|^2(x)\mathrm{d}x = \frac{1}{|M|}\int_M f(x) \, \mathrm{d}x + O(\hbar \mathrm{e}^{\Gamma|t|}) + O(\mathrm{e}^{-\gamma|t|})$$

- Remainder small if $\mathbf{0} << \boldsymbol{t} << \frac{\mathbf{1}}{\boldsymbol{\Gamma}} \mathbf{ln} \frac{\mathbf{1}}{\hbar}$, Ehrenfest time $T_E := \frac{1}{\Gamma} \ln \frac{1}{\hbar}$.
- Similar universality as in the classical system (mixing).

proof strategy: part 1, Egorov's theorem

We have

$$\langle \mathcal{U}(t)\psi, \mathrm{Op}[f]\mathcal{U}(t)\psi\rangle = \langle \psi, \mathcal{U}(t)^* \mathrm{Op}[f]\mathcal{U}(t)\psi\rangle$$

Theorem. [Bambusi, Graffi and Paul (99); Bouzouina and Robert (02)]
There exists a $k > 1$ such that for all $f \in C_0^\infty(T^*M)$

$$\|\mathcal{U}(t)^* \mathrm{Op}[f]\mathcal{U}(t) - \mathrm{Op}[f \circ \phi^{-t}]\|_{L^2} = O(\hbar \|f \circ \phi^{-t}\|_{C^k}) \ .$$

Remarks:

- Correspondence principle: for $\hbar \to 0$ we find quantum $\to$ classical.
- if ϕ^t is Anosov, then $\|f \circ \phi^{-t}\|_{C^k} = O(\mathrm{e}^{k\lambda|t|})$

Using Egorov's Theorem

$$\begin{aligned}\langle \mathcal{U}(t)\psi, \mathrm{Op}[f]\mathcal{U}(t)\psi\rangle &= \langle \psi, \mathcal{U}(t)^* \mathrm{Op}[f]\mathcal{U}(t)\psi\rangle \\ &= \langle \psi, \mathrm{Op}[f \circ \phi^{-t}]\psi\rangle + O(\hbar \| f \circ \phi^{-t}\|_{C^k}) \\ &= \int_\Lambda f \circ \phi^{-t} \rho(\psi) + O(\hbar \| p \circ \phi^{-t}\|_{C^k})\end{aligned}$$

So we have to understand how Λ and the density $\rho(\psi)$ on Λ are transported under the geodesic flow.

step II: mixing of submanifolds

Theorem. [Eskin McMullen 93, Sinai 95, Chernov 97, RS 05] *Assume $\phi^t : S^*M \to S^*M$ is Anosov, and $\Lambda \subset S^*M$ is a submanifold with $\dim\Lambda = \dim M$ and which is transversal to the stable foliation. Then there is a $\gamma' > 0$ such that for any C^1 density ρ on Λ and $f \in C^1(S^*M)$*

$$\int_\Lambda f\circ\phi^{-t}\rho = \int_{S^*M} f\,\mathrm{d}\mu \int_\Lambda \rho + O(\|f\|_{C^1}\|\rho\|_{C^1}\mathrm{e}^{-\gamma' t})$$

Remarks:

- This is a consequence of the mixing results by Dolgopyat (98) and Liverani (05)

- The idea of proof goes back to Margulis.

- The assumptions can be relaxed a bit, but some transversality is necessary as the example of a stable manifold shows.

Combining this result with the semiclassical estimates gives the proof for the Anosov case.

Localized states

Theorem. [RS 06] *Let* $\delta > 0$, $a_0 \in C^\infty(M)$ *and set*

$$a(\hbar, x) = \frac{1}{\hbar^{\frac{d\delta}{2}}} a_0\Big(\frac{x - x_0}{\hbar^\delta}\Big) \qquad \psi_0(x) = a(\hbar, x)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi(x)} \ .$$

Assume ϕ^t *is Anosov,* $\Lambda_\varphi \subset S^*M$, *and* Λ_φ *is transversal to the stable foliation. Then there exists constants* $\Gamma, \lambda \geq \gamma > 0$ *such that for all* $f \in C_0^\infty(T^*M)$ *and if* $\|\psi\|_{L^2} = 1$

$$\lim_{\hbar \to 0} \langle \mathcal{U}(t)\psi, \mathrm{Op}[f]\mathcal{U}(t)\psi\rangle = \begin{cases} f\big(\phi^{-t}(x_0, \mathrm{d}\varphi(x_0))\big) & \textit{if} \quad t << \frac{\delta}{\lambda}\ln\frac{1}{\hbar} \\ \int_{S^*M} f \ \mathrm{d}\mu & \textit{if} \quad \frac{\delta}{\gamma}\ln\frac{1}{\hbar} << t << \frac{1-\delta}{\Gamma}\ln\frac{1}{\hbar} \end{cases}$$

Time scales

- Ehrenfest time, breakdown of Egorov's theorem:

$$T_E \sim \frac{1}{\lambda} \ln \frac{1}{\hbar}$$

  for systems with positive Lyapunov exponents

- Heisenberg time, time scale to resolve spectrum:

$$T_H \sim \frac{1}{\hbar^{d-1}}$$

Summary

- Semiclassical propagation of wave-packets is driven by the propagation of (Lagrangian) submanifolds of phase space by the geodesic flow.

- Universality in geodesic flow on manifolds of negative curvature, mixing, implies a similar universality in wave propagation.

- The main open problem is to overcome the exponential proliferation in hyperbolic dynamics and to go beyond Ehrenfest time.