NONEXTENSIVE STATISTICAL MECHANICS AND ITS NONLINEAR DYNAMICAL FOUNDATIONS

Constantino Tsallis

Santa Fe Institute, New Mexico, USA

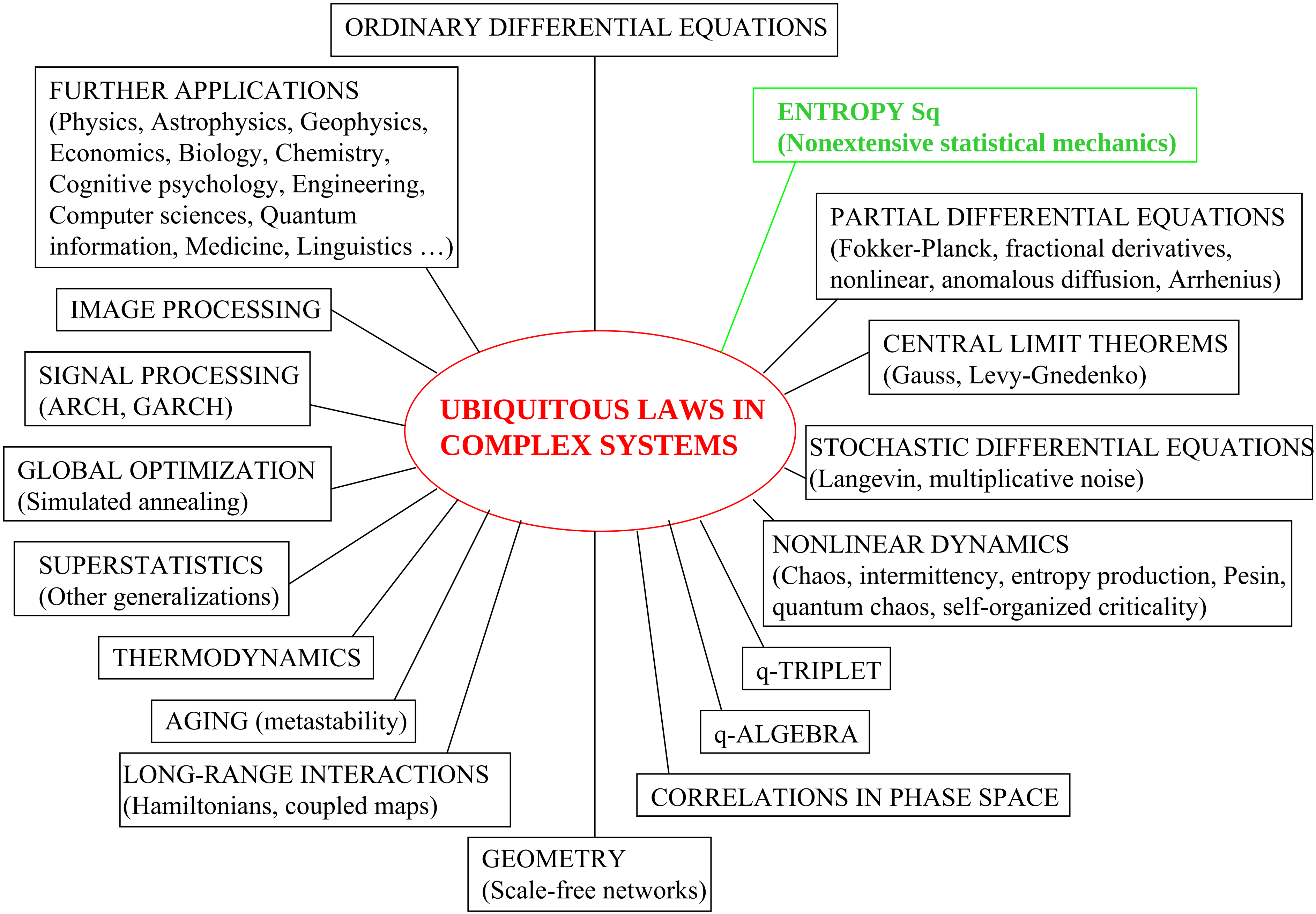
Centro Brasileiro de Pesquisas Fisicas, BRAZIL

C. T., M. Gell-Mann and Y. Sato, Proc Natl Acad Sci (USA) **102**, 15377 (2005)

L.G. Moyano, C. T. and M. Gell-Mann, Europhys Lett **73**, 813 (2006)

S. Umarov, C.T., M. Gell-Mann and S. Steinberg, cond-mat/0603593, 0606038, 0606040

Durham, July 2006



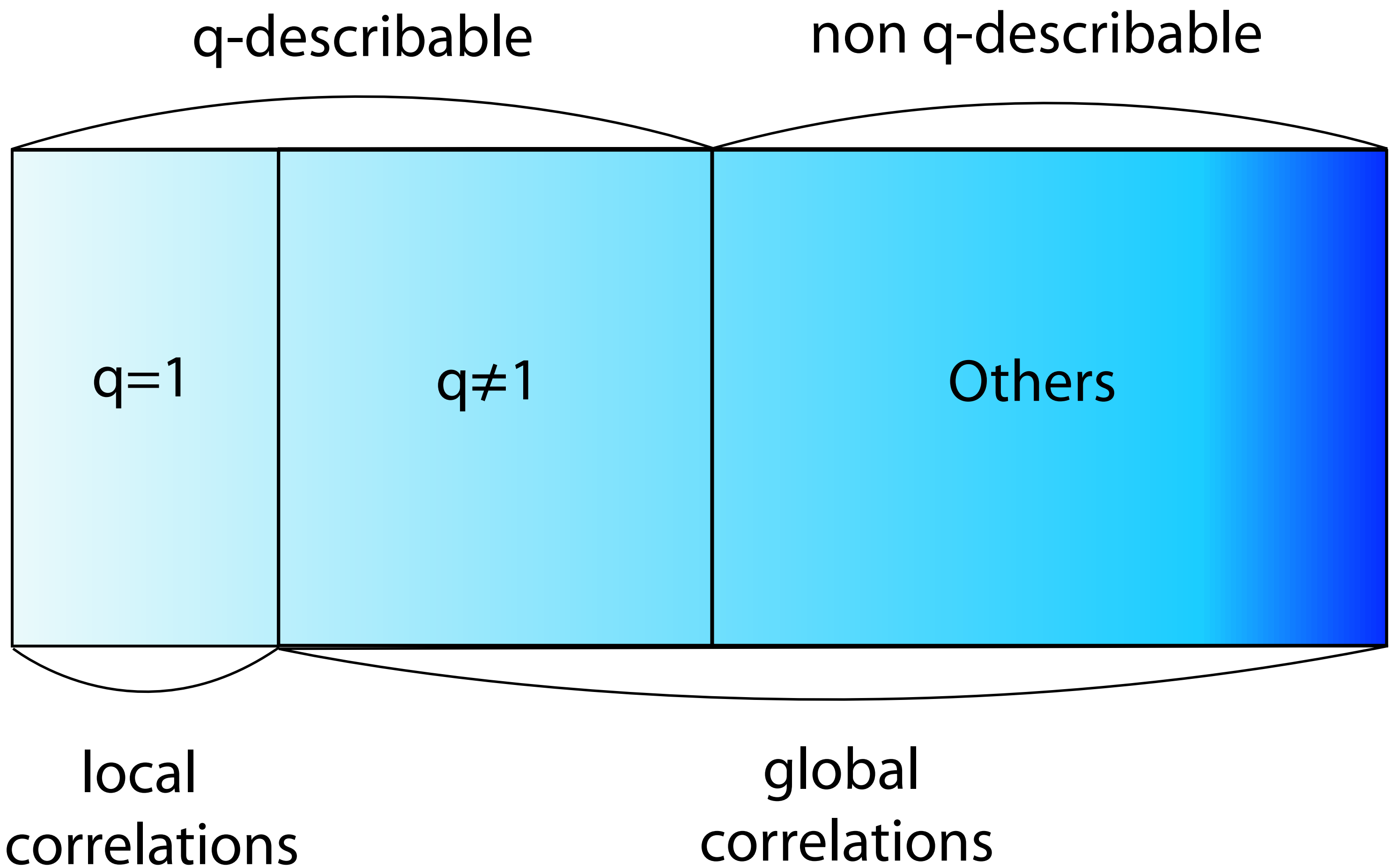
ORDINARY DIFFERENTIAL EQUATIONS
FURTHER APPLICATIONS (Physics, Astrophysics, Geophysics, Economics, Biology, Chemistry, Cognitive psychology, Engineering, Computer sciences, Quantum information, Medicine, Linguistics …)
ENTROPY Sq (Nonextensive statistical mechanics)
PARTIAL DIFFERENTIAL EQUATIONS (Fokker-Planck, fractional derivatives, nonlinear, anomalous diffusion, Arrhenius)
IMAGE PROCESSING
CENTRAL LIMIT THEOREMS (Gauss, Levy-Gnedenko)
SIGNAL PROCESSING (ARCH, GARCH)
UBIQUITOUS LAWS IN COMPLEX SYSTEMS
STOCHASTIC DIFFERENTIAL EQUATIONS (Langevin, multiplicative noise)
GLOBAL OPTIMIZATION (Simulated annealing)
NONLINEAR DYNAMICS (Chaos, intermittency, entropy production, Pesin, quantum chaos, self-organized criticality)
SUPERSTATISTICS (Other generalizations)
THERMODYNAMICS
q-TRIPLET
AGING (metastability)
q-ALGEBRA
LONG-RANGE INTERACTIONS (Hamiltonians, coupled maps)
CORRELATIONS IN PHASE SPACE
GEOMETRY (Scale-free networks)

ENTROPIC FORMS

	$p_i = \frac{1}{W} \quad (\forall i)$ equiprobability	$\forall p_i \; (0 \le p_i \le 1)$ $\left(\sum_{i=1}^{W} p_i = 1 \right)$
BG entropy $(q = 1)$	$k \ln W$	$-k \sum_{i=1}^{W} p_i \ln p_i$
Nonextensive entropy $(q \in \Re)$ $(q \ne 1)$	$k \frac{W^{1-q} - 1}{1-q}$	$k \frac{1 - \sum_{i=1}^{W} p_i^q}{q-1}$

Possible generalization of Boltzmann-Gibbs statistical mechanics

[C.T., J. Stat. Phys. **52**, 479 (1988)]



C.T., M. Gell-Mann and Y. Sato
Europhysics News **36** (6), 186 (2005)
[European Physical Society]

SANTOS THEOREM: RJV Santos, J Math Phys **38**, 4104 (1997)

(q - generalization of Shannon 1948 theorem)

IF $S(\{p_i\})$ *continuous function of* $\{p_i\}$

AND $S(p_i = 1/W, \forall i)$ *monotonically increases with* W

AND $\frac{S(A+B)}{k} = \frac{S(A)}{k} + \frac{S(B)}{k} + (1-q)\frac{S(A)}{k}\frac{S(B)}{k}$ $\quad (\text{with } p_{ij}^{A+B} = p_i^A p_j^B)$

AND $S(\{p_i\}) = S(p_L, p_M) + p_L^q\, S(\{p_l / p_L\}) + p_M^q\, S(\{p_m / p_M\})$ $\quad (\text{with } p_L + p_M = 1)$

THEN AND ONLY THEN

$$S(\{p_i\}) = k\frac{1-\sum_{i=1}^{W} p_i^q}{q-1} \qquad \left(q = 1 \Rightarrow S(\{p_i\}) = -k\sum_{i=1}^{W} p_i \ln p_i\right)$$

CE SHANNON (The Mathematical Theory of Communication):

"This theorem, and the assumptions required for its proof, are in no way necessary for the present theory. It is given chiefly to lend a certain plausibility to some of our later definitions. The real justification of these definitions, however, will reside in their implications.

ABE THEOREM: S Abe, Phys Lett A **271**, 74 (2000)

(q - generalization of Khinchin 1953 theorem)

IF $S(\{p_i\})$ *continuous function of* $\{p_i\}$

AND $S(p_i = 1/W,\ \forall i)$ *monotonically increases with* W

AND $S(p_{1,} p_2, ..., p_W, 0) = S(p_{1,} p_2, ..., p_W)$

AND $\frac{S(A+B)}{k} = \frac{S(A)}{k} + \frac{S(B|A)}{k} + (1-q)\frac{S(A)}{k}\frac{S(B|A)}{k}$

THEN AND ONLY THEN

$$S(\{p_i\}) = k\frac{1-\sum_{i=1}^{W} p_i{}^q}{q-1} \quad \left(q=1 \Rightarrow S(\{p_i\}) = -k\sum_{i=1}^{W} p_i \ln p_i\right)$$

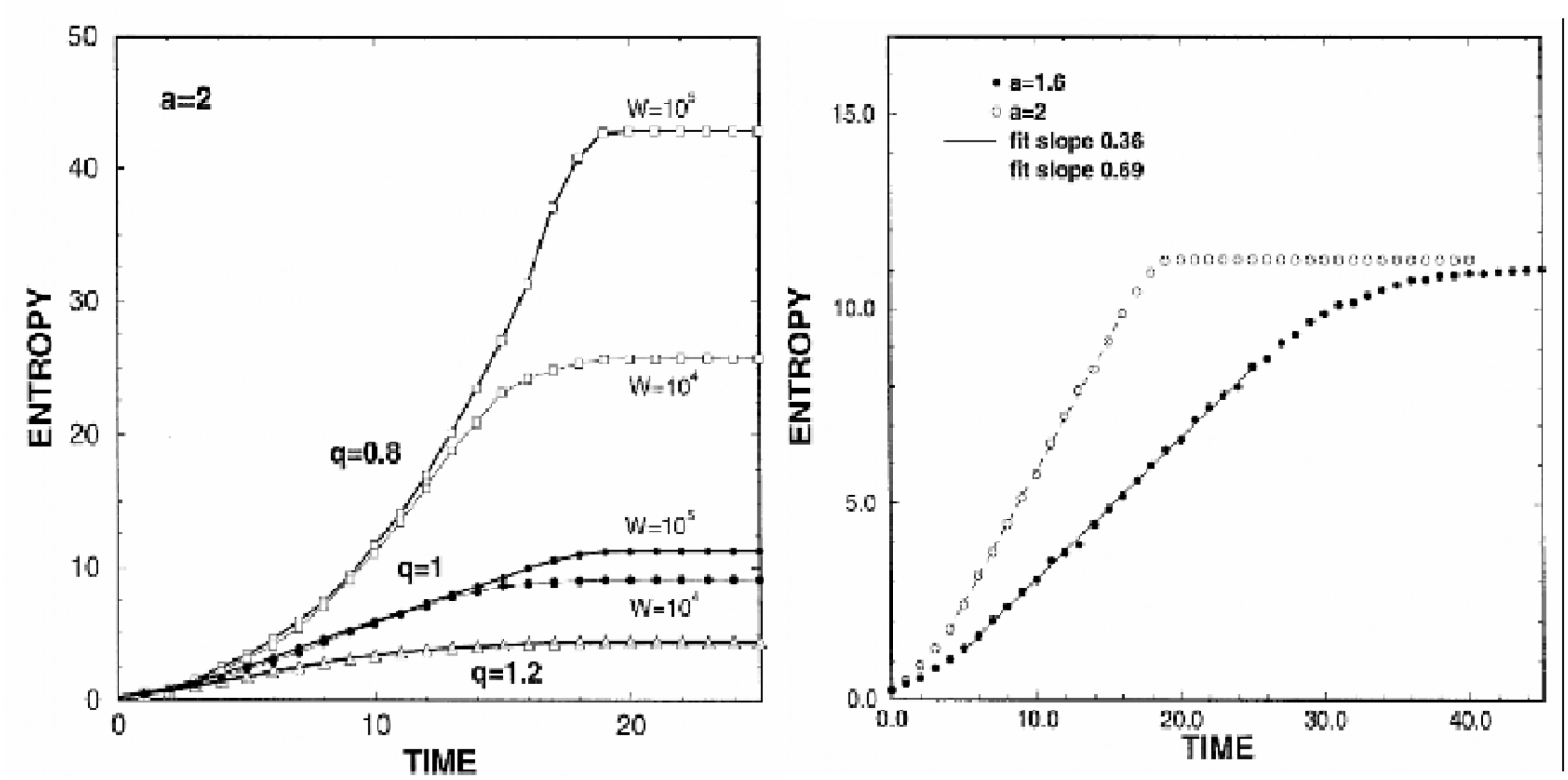
The possibility of such theorem was conjectured by AR Plastino and A Plastino (1996, 1999).

$$S_q(N,t)\ versus\ \ t$$

LOGISTIC MAP:

$$x_{t+1} = 1 - a\,x_t^2 \quad (0 \le a \le 2;\ -1 \le x_t \le 1;\ t = 0,1,2,\ldots)$$

(strong chaos, i.e., **positive** Lyapunov exponent)



V. Latora, M. Baranger, A. Rapisarda and C. T., Phys. Lett. A **273**, 97 (2000)

We verify

$$K_1 = \lambda_1 \quad (Pesin-like\ identity)$$

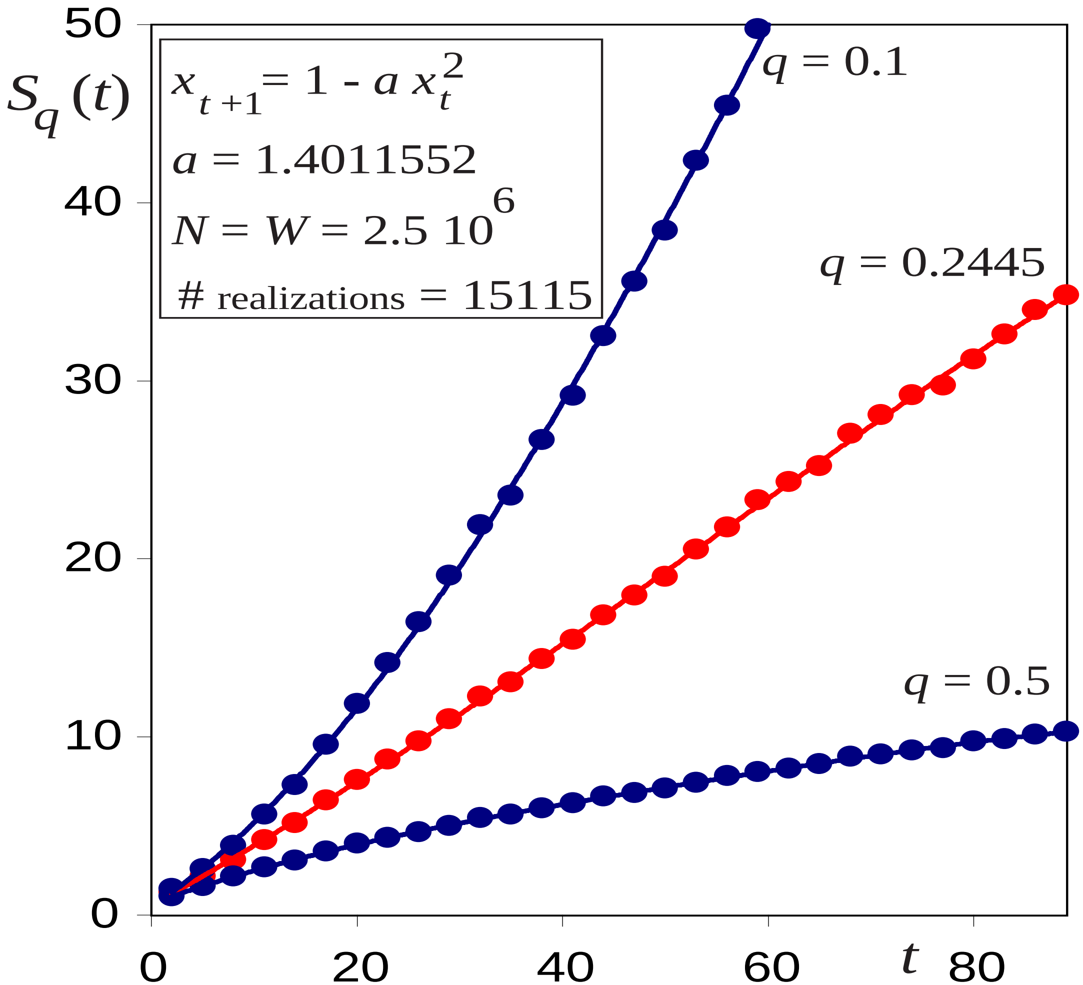
where

$$K_1 \equiv \lim_{t\to\infty} \frac{S_1(t)}{t}$$

and

$$\xi(t) \equiv \lim_{\Delta x(0)\to 0} \frac{\Delta x(t)}{\Delta x(0)} = e^{\lambda_1\, t}$$

(weak chaos, i.e., **zero** Lyapunov exponent)



C. T. , A.R. Plastino and W.-M. Zheng, Chaos, Solitons & Fractals **8**, 885 (1997)
M.L. Lyra and C. T. , Phys. Rev. Lett. **80**, 53 (1998)
V. Latora, M. Baranger, A. Rapisarda and C. T. , Phys. Lett. A **273**, 97 (2000)
E.P. Borges, C. T. , G.F.J. Ananos and P.M.C. Oliveira, Phys. Rev. Lett. **89**, 254103 (2002)
F. Baldovin and A. Robledo, Phys. Rev. E **66**, R045104 (2002) and **69**, R045202 (2004)
G.F.J. Ananos and C. T. , Phys. Rev. Lett. **93**, 020601 (2004)
E. Mayoral and A. Robledo, Phys. Rev. E **72**, 026209 (2005), and references therein

We verify

$$K_q = \lambda_q \quad (q-generalized \;\; Pesin-like \; identity)$$

where

$$K_q \equiv \lim_{t\to\infty} \sup\left\{\frac{S_q(t)}{t}\right\}$$

and

$$\xi(t) \equiv \sup\left\{\lim_{\Delta x(0)\to 0} \frac{\Delta x(t)}{\Delta x(0)}\right\} = e_q^{\lambda_q\, t}$$

with

$$e_q^z \equiv [1+(1-q)\, z]^{\frac{1}{1-q}} \qquad \left(e_1^z = e^z\right)$$

THE CASATI-PROSEN TRIANGLE MAP:

G. Casati and T. Prosen,
Phys. Rev. Lett. **83**, 4729 (1999) and **85**, 4261 (2000)

"While exponential instability is sufficient for a meaningful statistical description, it is not known whether or not it is also necessary."

$$y_{t+1} = y_t + \alpha \ \mathrm{sgn}(x_t) + \beta \quad (\mathrm{mod}\, 2)$$

$$x_{t+1} = x_t + y_{t+1} \quad (\mathrm{mod}\, 2)$$

(α and β independent irrationals)

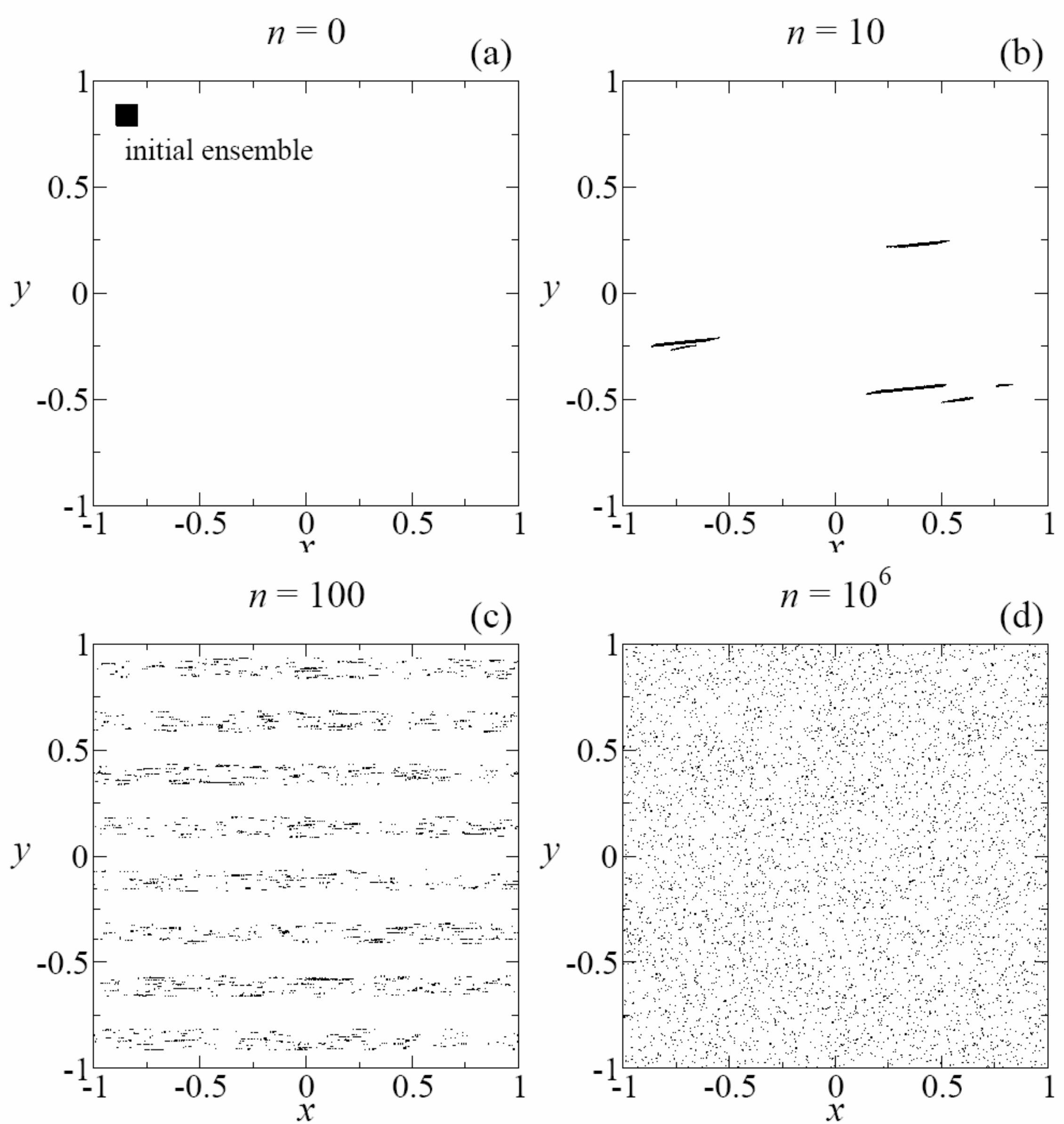
e.g., $(\alpha,\beta)=\left((1/2)(\sqrt{5}-1)-(1/e),\ (1/2)(\sqrt{5}-1)+(1/e)\right)$

This map is conservative, mixing, ergodic
and nevertheless with zero Lyapunov exponent!

Furthermore $\xi \equiv \lim_{\Delta X(0)\to 0} \frac{\Delta X(t)}{\Delta X(0)} \propto t$

CASATI-PROSEN TRIANGLE MAP [Casati and Prosen, Phys Rev Lett **83**, 4729 (1999) and **85**, 4261 (2000)]

(two-dimensional, conservative, mixing, ergodic, vanishing maximal Lyapunov exponent)



G. Casati, C. T. and F. Baldovin, Europhys. Lett. **72**, 355 (2005)

NONEXTENSIVITY OF THE CASATI-PROSEN MAP:

Answer to the above equation:

[G. Casati, C.T. and F. Baldovin, Europhys Lett **72**, 355 (2005)]

It is not necessary: a meaningful statistical description is possible with zero Lyapunov exponent!

[Essentially because an integrable system has zero Lyapunov exponent but the opposite is not true]

In general, $\xi = [1+(1\text{-q})\lambda_q t]^{1/(1-q)}$

hence, $\xi \propto t \Rightarrow q = 0$

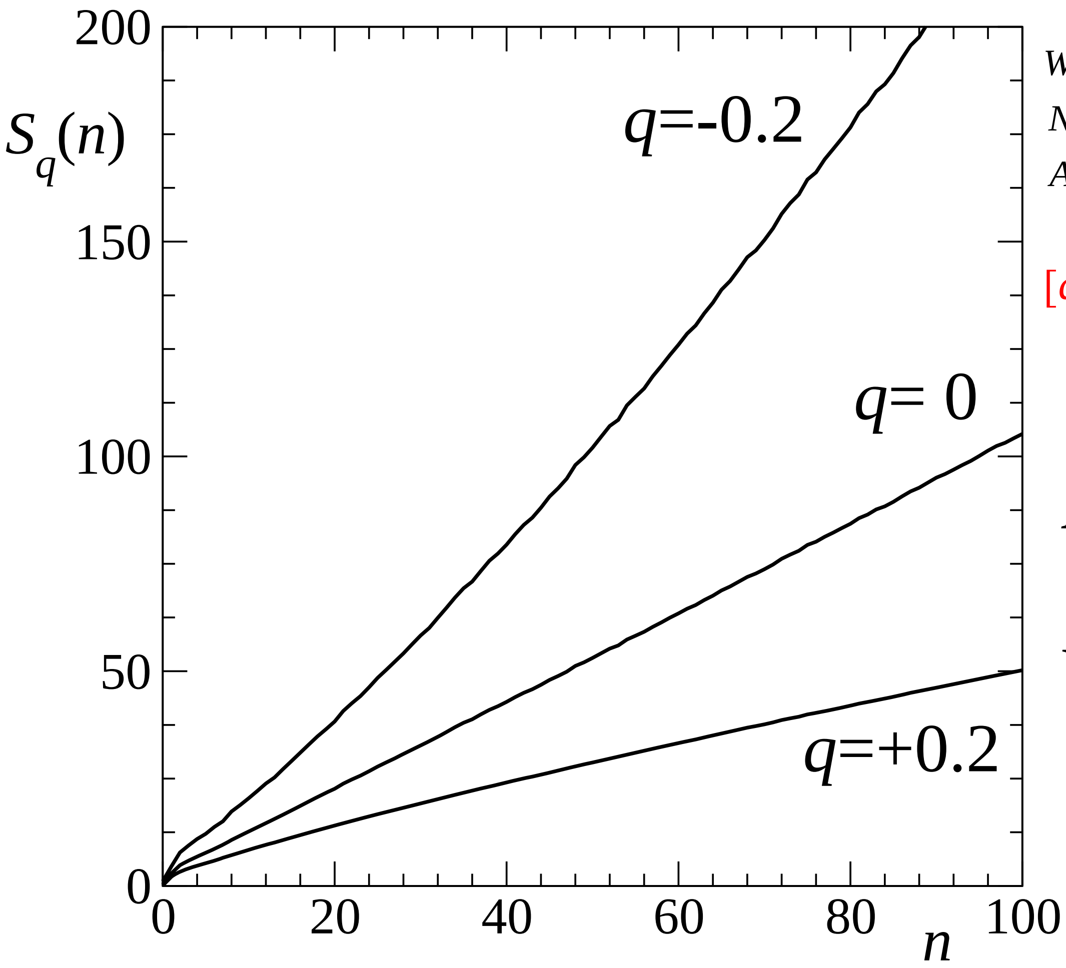
Consistently, we expect

$$(i)\ S_q(t) \equiv \frac{1-\sum_{i=1}^{W}[p_i(t)]^q}{q-1} \propto t \text{ only for } q=0$$

$$(ii) K_q \equiv \lim_{t\to\infty} \frac{S_q(t)}{t} = \lambda_q \qquad \text{for } q=0$$

CASATI-PROSEN TRIANGLE MAP [Casati and Prosen, Phys Rev Lett **83**, 4729 (1999) and **85**, 4261 (2000)]

(two-dimensional, conservative, mixing, ergodic, vanishing maximal Lyapunov exponent)



$W = 4000 \times 4000$ *cells*

$N = 1000$ *initial conditions randomly chosen in one cell*

Average done over 100 *initial cells*

[$q = 0 \;\rightarrow\;$ *linear correlation* $= 0.99993$]

Also $\xi = e_0^{\lambda_0 t}$

with $\lambda_0 = \lim_{n\to\infty} \frac{S_0(n)}{n} = 1$

q - generalization of Pesin (- like) theorem

G. Casati, C. T. and F. Baldovin, Europhys. Lett. **72**, 355 (2005)

$$S_q(N,t) \text{ versus } N$$

HYBRID PASCAL - LEIBNITZ TRIANGLE

(N = 0) $$1 \times \frac{1}{1}$$

(N = 1) $$1 \times \frac{1}{2} \quad 1 \times \frac{1}{2}$$

(N = 2) $$1 \times \frac{1}{3} \quad 2 \times \frac{1}{6} \quad 1 \times \frac{1}{3}$$

(N = 3) $$1 \times \frac{1}{4} \quad 3 \times \frac{1}{12} \quad 3 \times \frac{1}{12} \quad 1 \times \frac{1}{4}$$

(N = 4) $$1 \times \frac{1}{5} \quad 4 \times \frac{1}{20} \quad 6 \times \frac{1}{30} \quad 4 \times \frac{1}{20} \quad 1 \times \frac{1}{5}$$

(N = 5) $$1 \times \frac{1}{6} \quad 5 \times \frac{1}{30} \quad 10 \times \frac{1}{60} \quad 10 \times \frac{1}{60} \quad 5 \times \frac{1}{30} \quad 1 \times \frac{1}{6}$$

$$\Sigma = 1 \quad (\forall \; N)$$

Blaise **Pascal** (1623-1662)
Gottfried Wilhelm **Leibnitz** (1646-1716)
Daniel **Bernoulli** (1700-1782)

$(N=2)$

A \ B	1	2	
1	$p^2+\kappa$	$p(1-p)-\kappa$	p
2	$p(1-p)-\kappa$	$(1-p)^2+\kappa$	1- p
	p	1- p	1

EQUIVALENTLY:

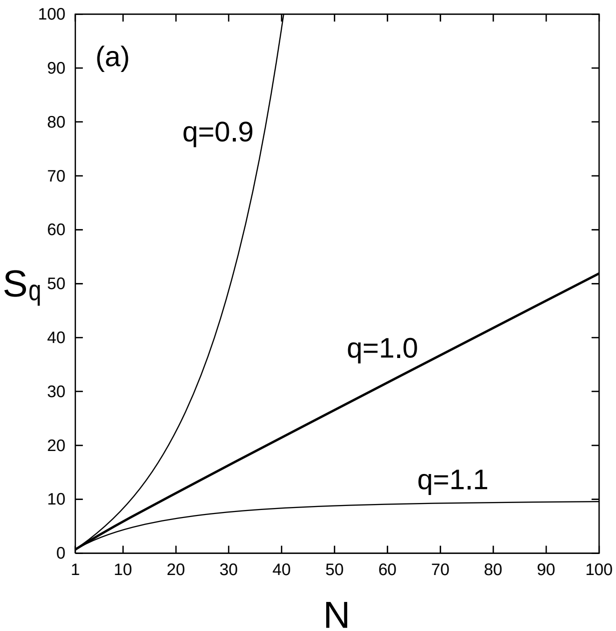
$(N=0)$ 1×1

$(N=1)$ $1\times p$ $1\times(1-p)$

$(N=2)$ $1\times[p^2+\kappa]$ $2\times[p(1-p)-\kappa]$ $1\times[(1-p)^2+\kappa]$

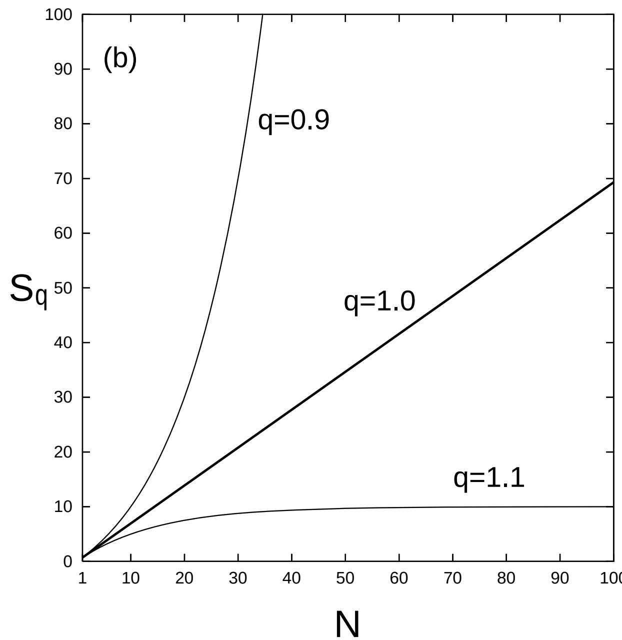
$q = 1$ SYSTEMS

i.e., such that $S_1(N) \propto N \quad (N \to \infty)$



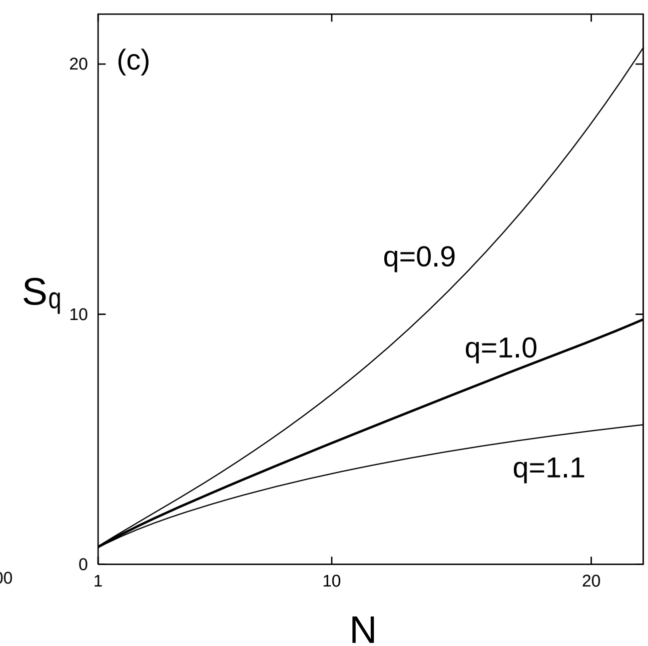
Leibnitz triangle

$$\left(p_{N,0} = \frac{1}{N+1} \right)$$



N independent coins

$$\left(\begin{array}{l} p_{N,0} = p^{N} \\ \\ \text{with } p = 1/2 \end{array} \right)$$



Stretched exponential

$$\left(\begin{array}{l} p_{N,0} = p^{N^{\alpha}} \\ \\ \text{with } p = \alpha = 1/2 \end{array} \right)$$

(All three examples **strictly** satisfy the Leibnitz rule)

C.T., M. Gell-Mann and Y. Sato
Proc Natl Acad Sc USA **102**, 15377 (2005)

Asymptotically scale-invariant (d=2)

$(N=0)$	1				
$(N=1)$	$1/2$	$1/2$			
$(N=2)$	$1/3$	$1/6$	$1/3$		
$(N=3)$	$3/8$	$5/48$	$5/48$	0	
$(N=4)$	$2/5$	$3/40$	$1/20$	0	0

d+1

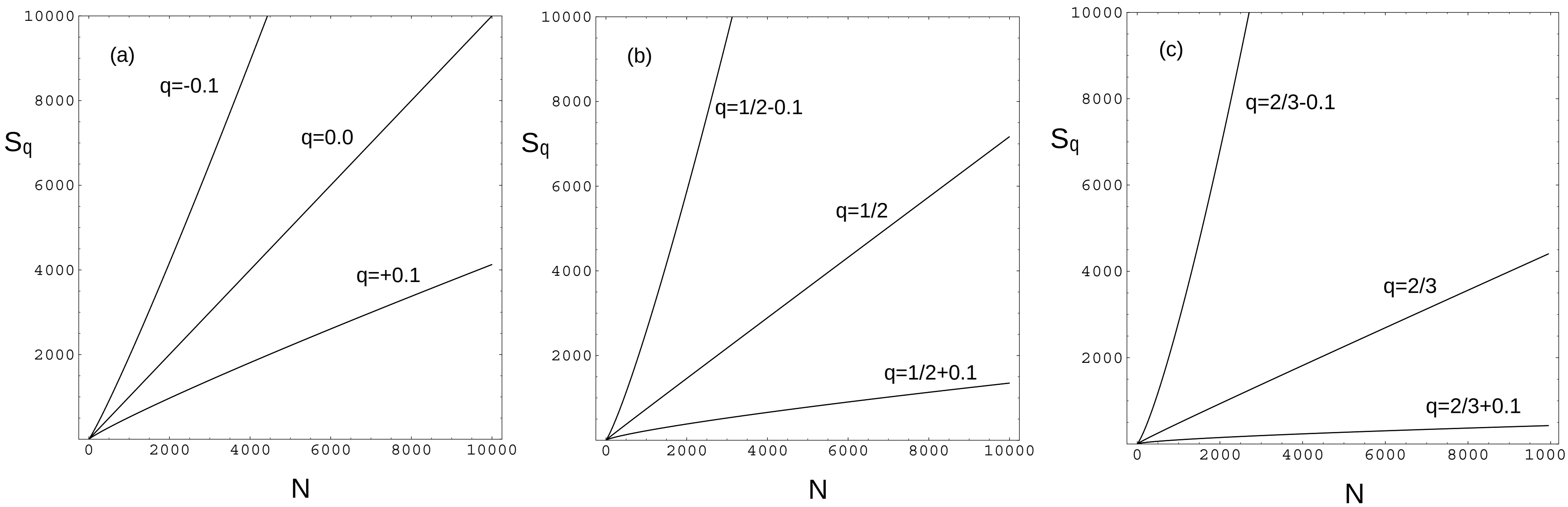
(It asymptotically satisfies the Leibnitz rule)

C.T., M. Gell-Mann and Y. Sato
Proc Natl Acad Sc USA **102**, 15377 (2005)

$q \neq 1$ SYSTEMS

i.e., such that $S_q(N) \propto N \quad (N \to \infty)$

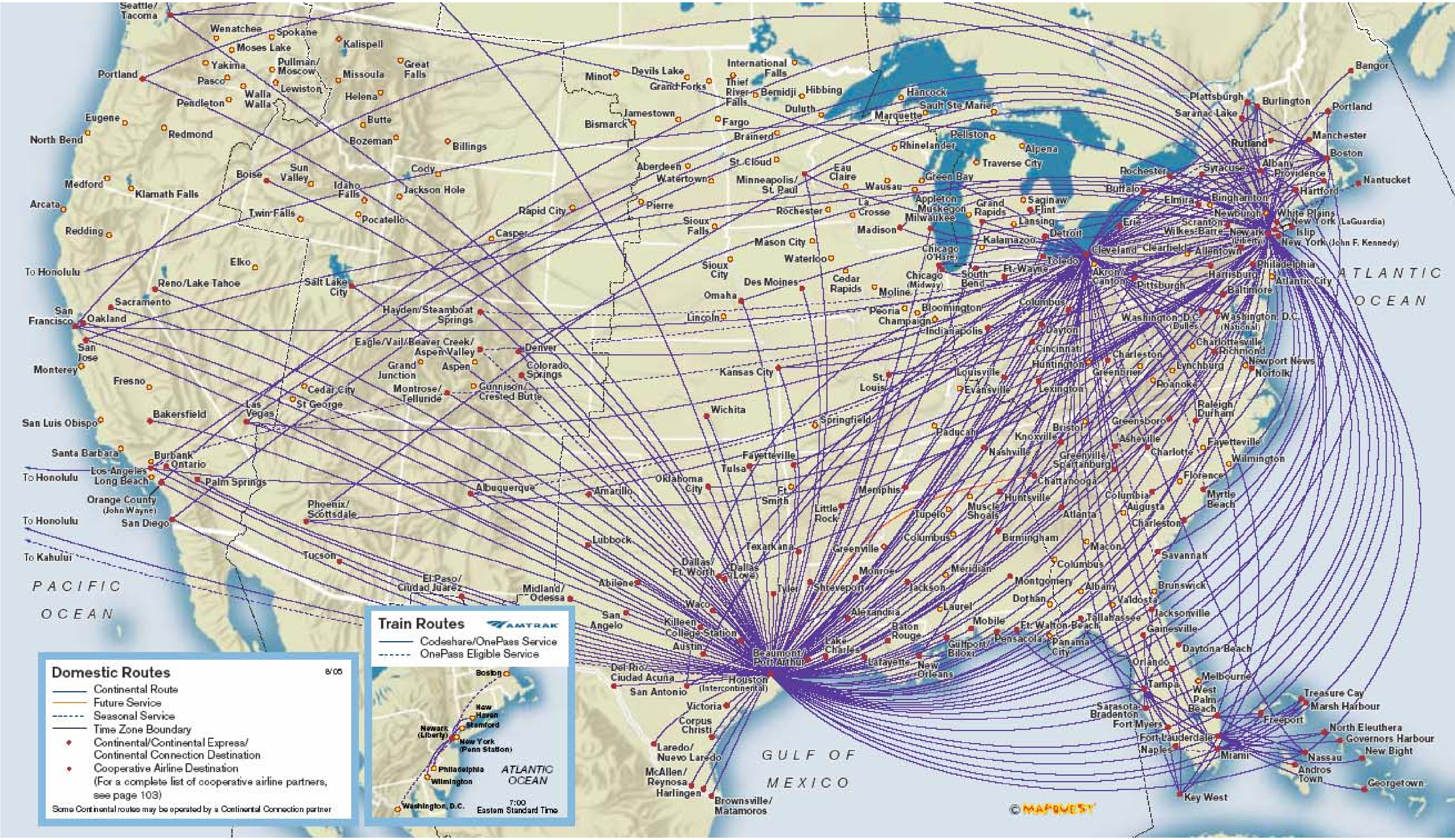
$(d = 1)$ $(d = 2)$ $(d = 3)$



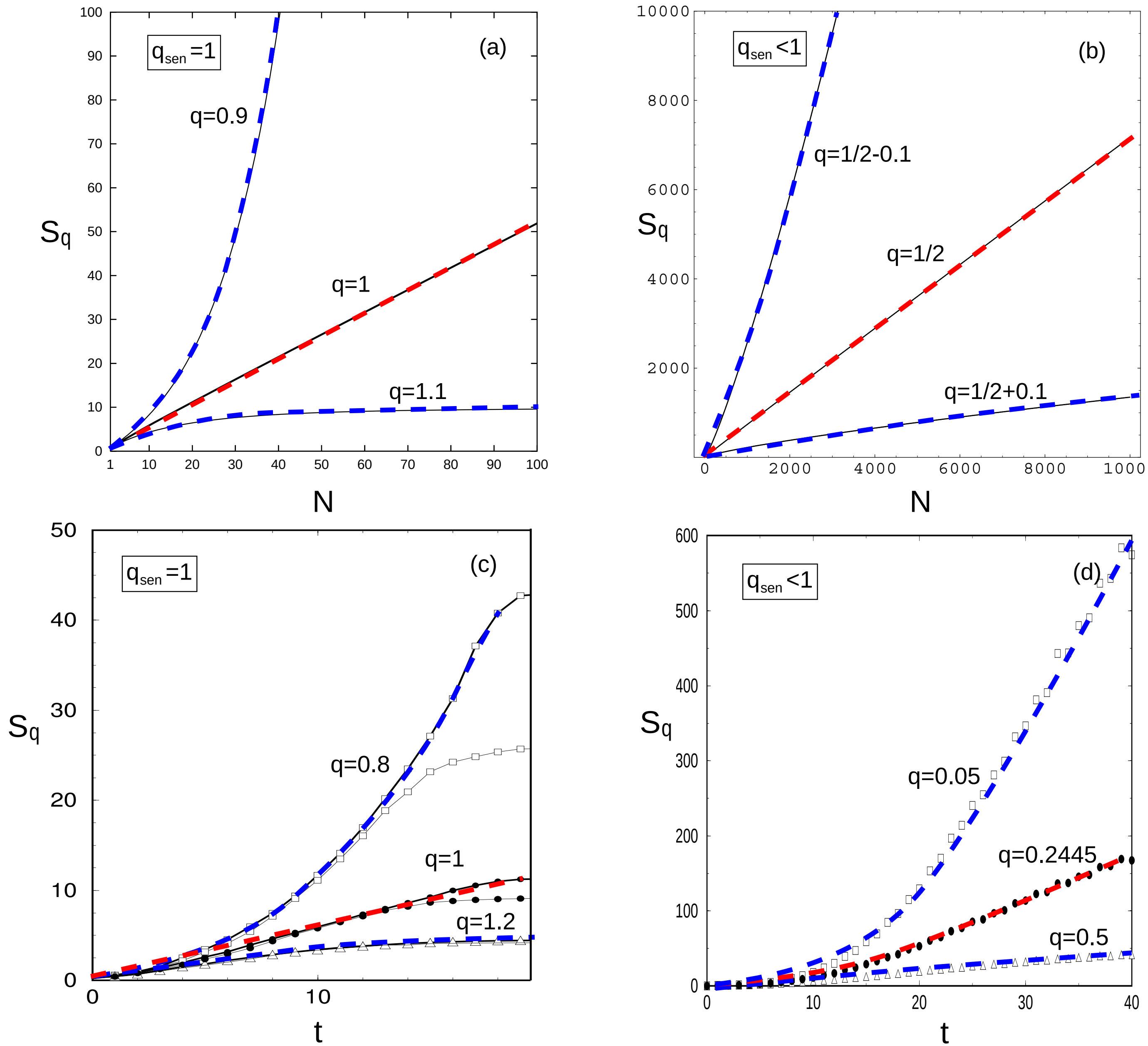
$$q = 1 - \frac{1}{d}$$

(All three examples **asymptotically** satisfy the **Leibnitz rule**)

C.T., M. Gell-Mann and Y. Sato
Proc Natl Acad Sc USA **102**, 15377 (2005)



Continental Airlines



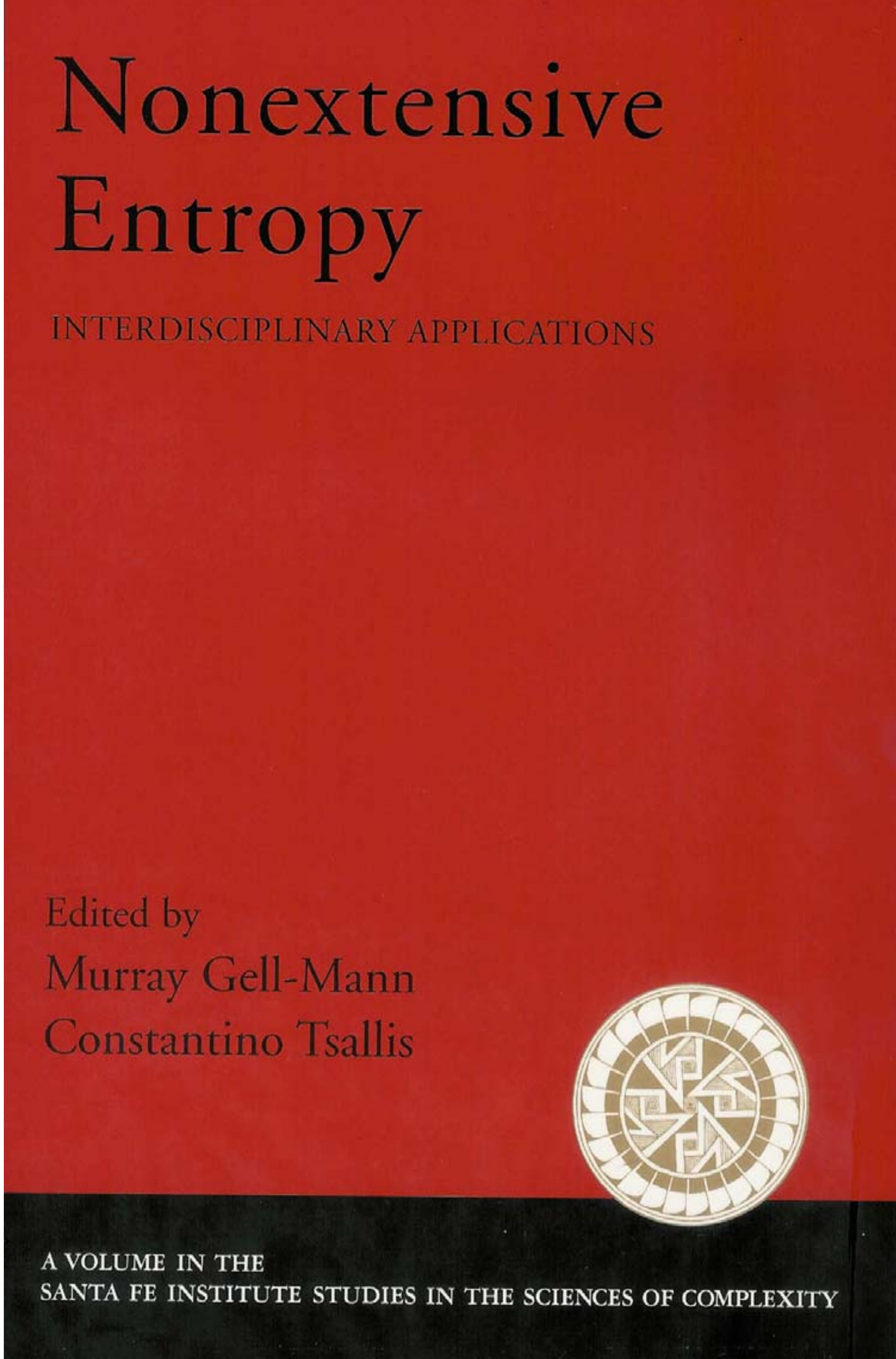
C.T., M. Gell-Mann and Y. Sato
Europhysics News **36** (6), 186 (2005) [European Physical Society]

If A and B are independent,

i.e., if $p_{ij}^{A+B} = p_i^{A}\, p_j^{B}$,

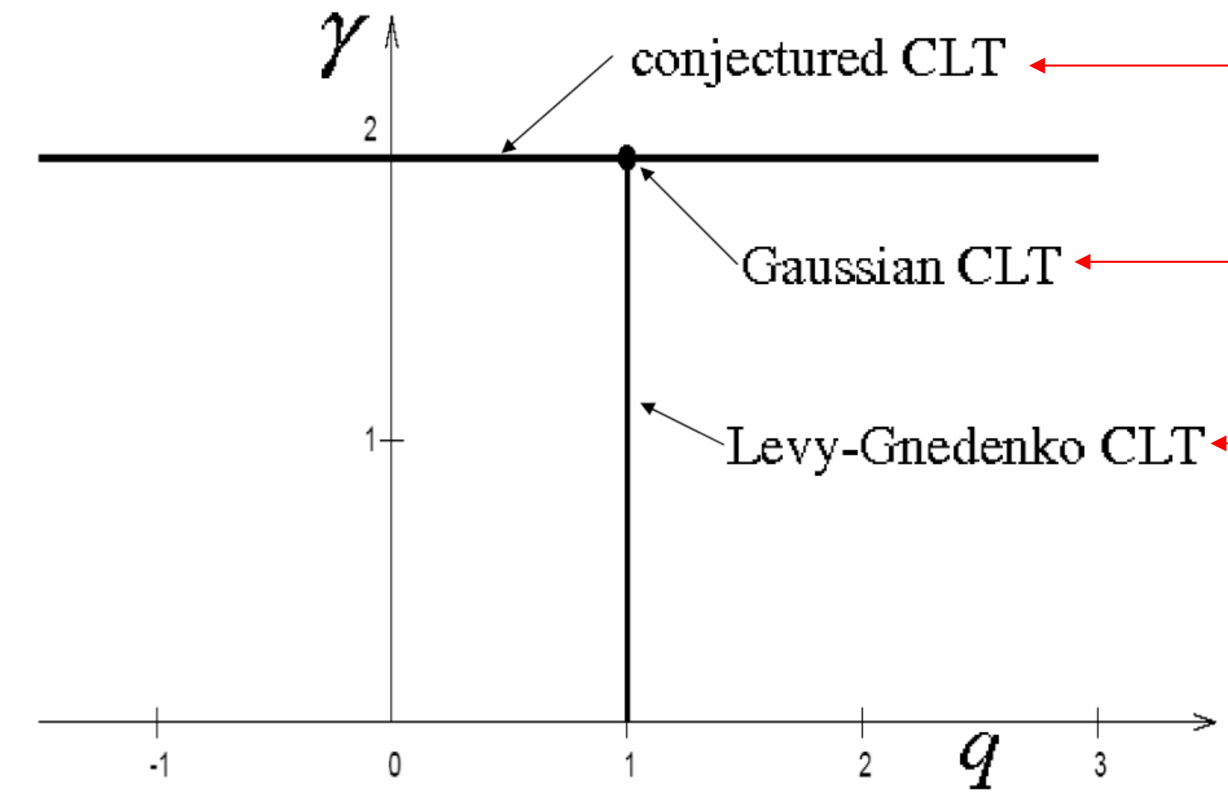
then

$$S_q(A+B) = S_q(A) + S_q(B) + \frac{1-q}{k_B}\, S_q(A)\, S_q(B)$$



q - CENTRAL LIMIT THEOREM: (conjecture)

$$\frac{\partial\, p(x,t)}{\partial t} = D\,\frac{\partial^{\gamma}[p(x,t)]^{2-q}}{\partial |x|^{\gamma}} \quad (0 < \gamma \le 2;\; q < 3)$$



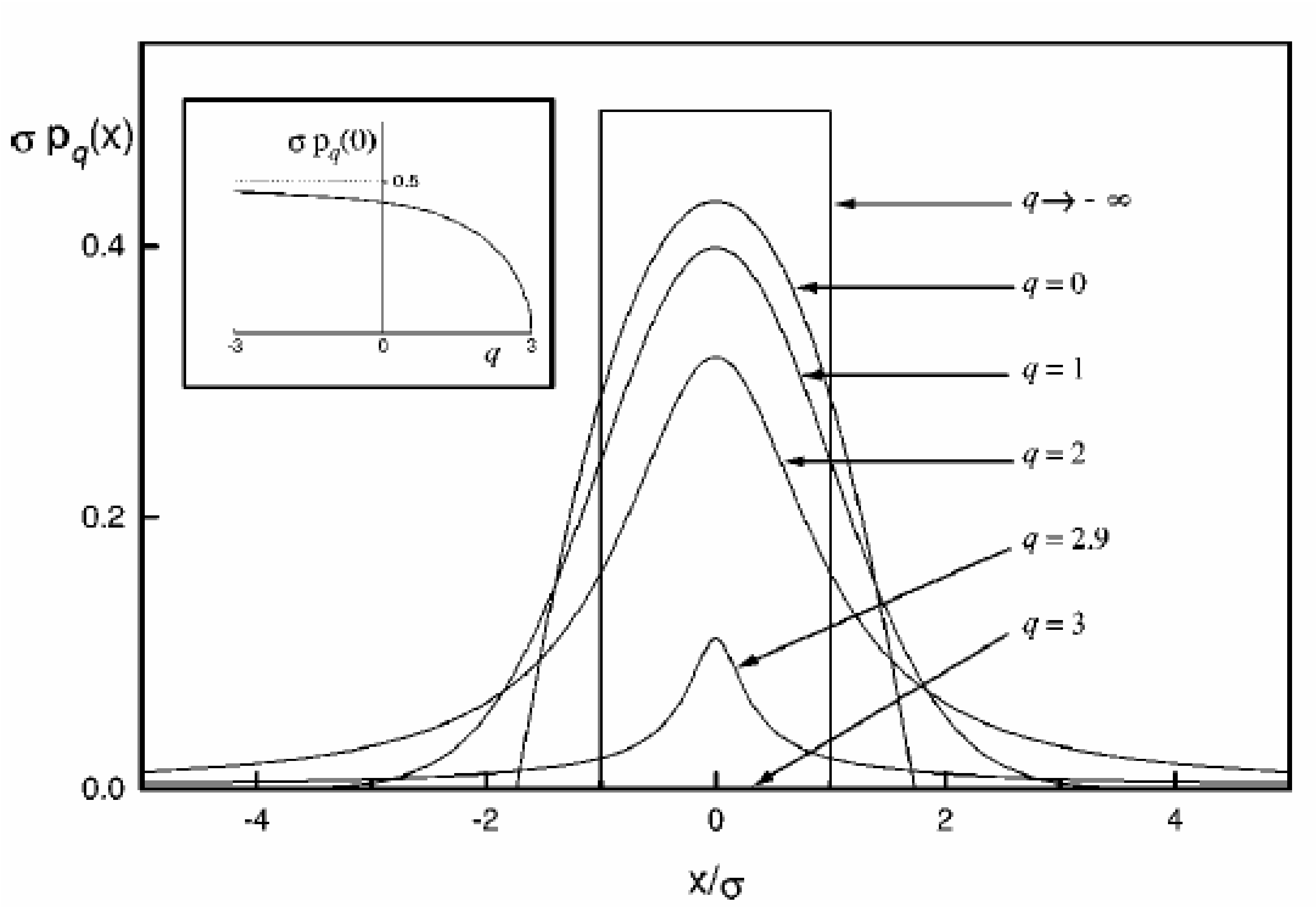
globally correlated variables;
finite q-variance;
q-Gaussian attractor

independent variables;
finite variance;
Gaussian attractor

independent variables;
divergent variance;
Levy attractor

C.T., Milan J. Math. **73**, 145 (2005)

q-GAUSSIANS:



D. Prato and C. T, Phys Rev E **60**, 2398 (1999)

q - CENTRAL LIMIT THEOREM (q-product and de Moivre-Laplace theorem):

The q- product is defined as follows:

$$x \otimes_q y \equiv \left[x^{1-q} + y^{1-q} - 1 \right]^{\frac{1}{1-q}}$$

$$Properties:$$

$$i) \quad x \otimes_1 y = x\, y$$

$$ii) \quad \ln_q (x \otimes_q y) = \ln_q x + \ln_q y$$

$$[whereas \;\; \ln_q (x\, y) = \ln_q x + \ln_q y + (1-q)(\ln_q x)(\ln_q y)]$$

[L. Nivanen, A. Le Mehaute and Q.A. Wang, Rep. Math. Phys. **52**, 437 (2003); E.P. Borges, Physica A **340**, 95 (2004)]

The de Moivre-Laplace theorem can be constructed with

$$i) \quad p_{N,0} = \left(\frac{1}{2} \right)^N$$

$$and$$

$$ii) \; Leibnitz \;\; rule$$

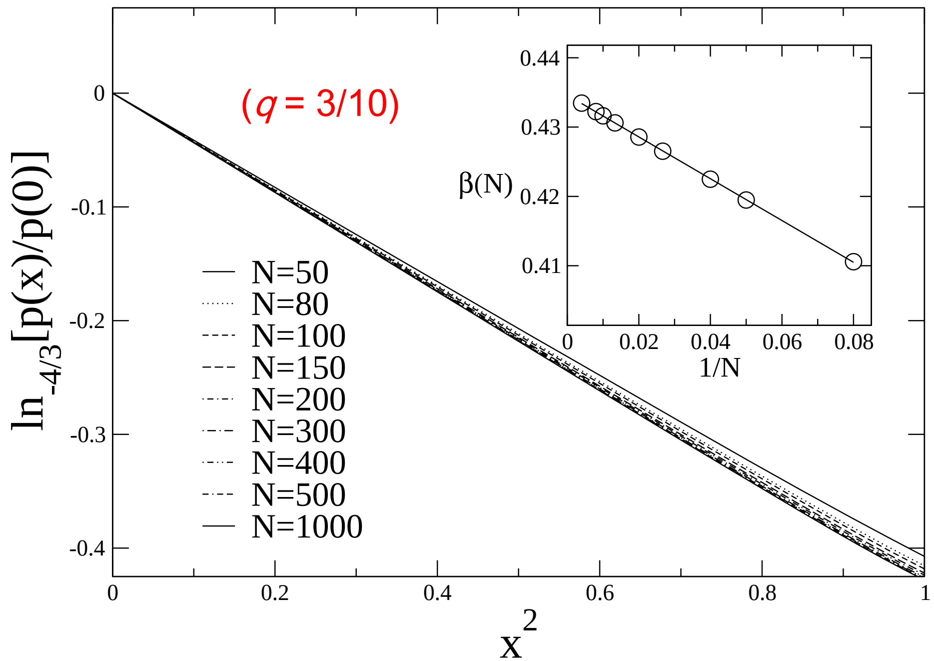
q - CENTRAL LIMIT THEOREM: (numerical indications)

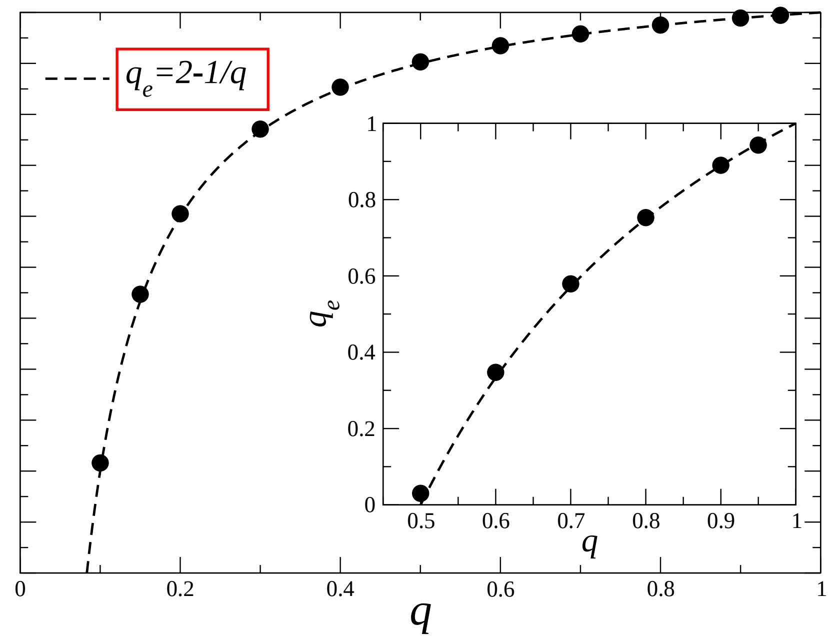
We q – generalize the de Moivre – Laplace theorem with

$$\frac{1}{p_{N,0}} = \left(\frac{1}{p}\right) \otimes_q \left(\frac{1}{p}\right) \otimes_q \dots \left(\frac{1}{p}\right) \quad (N \ terms)$$

i.e.,

$$p_{N,0} = \left[N\, p^{q-1} - (N-1)\right]^{\frac{1}{q-1}} \quad (with\ p = 1/2)$$





[Hence q → 2 – q (additive duality) and q → 1/q (multiplicative duality) are involved]

L.G. Moyano, C. T. and M. Gell-Mann, Europhys. Lett. **73**, 813 (2006)

q- GENERALIZED CENTRAL LIMIT THEOREM: (mathematical proof)

S. Umarov, C.T. and S. Steinberg [cond-mat/0603593]

q-Fourier transform:

$$F_q[f](\xi) \equiv \int_{-\infty}^{\infty} e_q^{ix\xi} \otimes_q f(x)\,dx = \int_{-\infty}^{\infty} e_q^{\frac{ix\xi}{[f(x)]^{1-q}}} f(x)\,dx \qquad \text{(nonlinear!)}$$

q-correlation:

Two random variables X [*with density* $f_X(x)$] *and* Y [*with density* $f_Y(y)$]

are said q*-correlated if*

$$F_q[X+Y](\xi) = F_q[X](\xi) \otimes_q F_q[Y](\xi) \,,$$

i.e., if

$$\int_{-\infty}^{\infty} dz\, e_q^{iz\xi} \otimes_q f_{X+Y}(z) = \left[\int_{-\infty}^{\infty} dx\, e_q^{ix\xi} \otimes_q f_X(x)\right] \otimes_q \left[\int_{-\infty}^{\infty} dy\, e_q^{iy\xi} \otimes_q f_Y(y)\right],$$

with $f_{X+Y}(z) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, h(x,y)\, \delta(x+y-z) = \int_{-\infty}^{\infty} dx\, h(x, z-x) = \int_{-\infty}^{\infty} dy\, h(z-y, y)$

where $h(x,y)$ *is the joint density.*

$$\begin{pmatrix} q\text{-correlation means} & \text{independence} & \text{if } q=1\,,\ \text{i.e.,} \ \ h(x,y) = f_X(x)\, f_Y(y) \\ & \text{global correlation} & \text{if } q\neq 1\,,\ \text{hence } \ h(x,y) \neq f_X(x)\, f_Y(y) \end{pmatrix}$$

Closure:

The q-Fourier transform of a q-Gaussian is a z(q)-Gaussian with

$$z(q) = \frac{1+q}{3-q} \in (-\infty, 3)$$

Iteration:

$$q_n \equiv z_n(q) \equiv z(z_{n-1}(q)) = \frac{2q+n(1-q)}{2+n(1-q)} \quad (n=0,\ \pm 1,\ \pm 2, \ldots;\ q_0 = q)$$

(the same as in R.S. Mendes and C.T. [Phys Lett A 285, 273 (2005)] when calculating marginal probabilities!)

hence

(i) $q_n(1) = 1\ (\forall n), \quad q_{\pm\infty}(q) = 1\ (\forall q),$

(ii) $q_{n-1} = 2 - \frac{1}{q_{n+1}}$,

(the same as in L.G. Moyano, C.T. and M. Gell-Mann (2005)!)

(the same as in A. Robledo [Physica D 193, 153 (2004)] for pitchfork and tangent bifurcations!)

(iii) $n = 2m = 0,\ \pm 2,\ \pm 4, \ldots$ *yields* $q_{(m)} \equiv q_{2m} = \frac{q+m(1-q)}{1+m(1-q)}$

(the same obtained in C.T., M. Gell-Mann and Y. Sato [Proc Natl Acad Sci (USA) 102, 15377 (2005)], by combining *only* additive and multiplicative dualities, and which was conjectured to be a possible explanation for the NASA-detected q-triangle for m = 0, ±1!)

Generic pitchfork bifurcations:

$$x_{t+1} = x_t + b\ sign(x_t)\,|\,x_t\,|^z\ (z > 1;\ b > 0)$$

Generic tangent bifurcations:

$$x_{t+1} = x_t + b\,|\,x_t\,|^z\ (z > 1;\ b > 0)$$

The fixed point map is a q-exponential with

$$q = z$$

and the sensitivity to the initial conditions is a q_{sen}-exponential with

$$q_{sen} = 2 - \frac{1}{q}$$

Example: The ς – logistic family of maps

$$x_{t+1} = 1 - a\,|\,x_t\,|^\varsigma\ (\varsigma > 1;\ 0 \le a \le 2;\ \varsigma > 1)$$

has

$z = 3$ *for pitchfork bifurcations* $(\forall \varsigma)$, *hence* $q = 3$ *and* $q_{sen} = \frac{5}{3}$;

$z = 2$ *for tangent bifurcations* $(\forall \varsigma)$, *hence* $q = 2$ *and* $q_{sen} = \frac{3}{2}$.

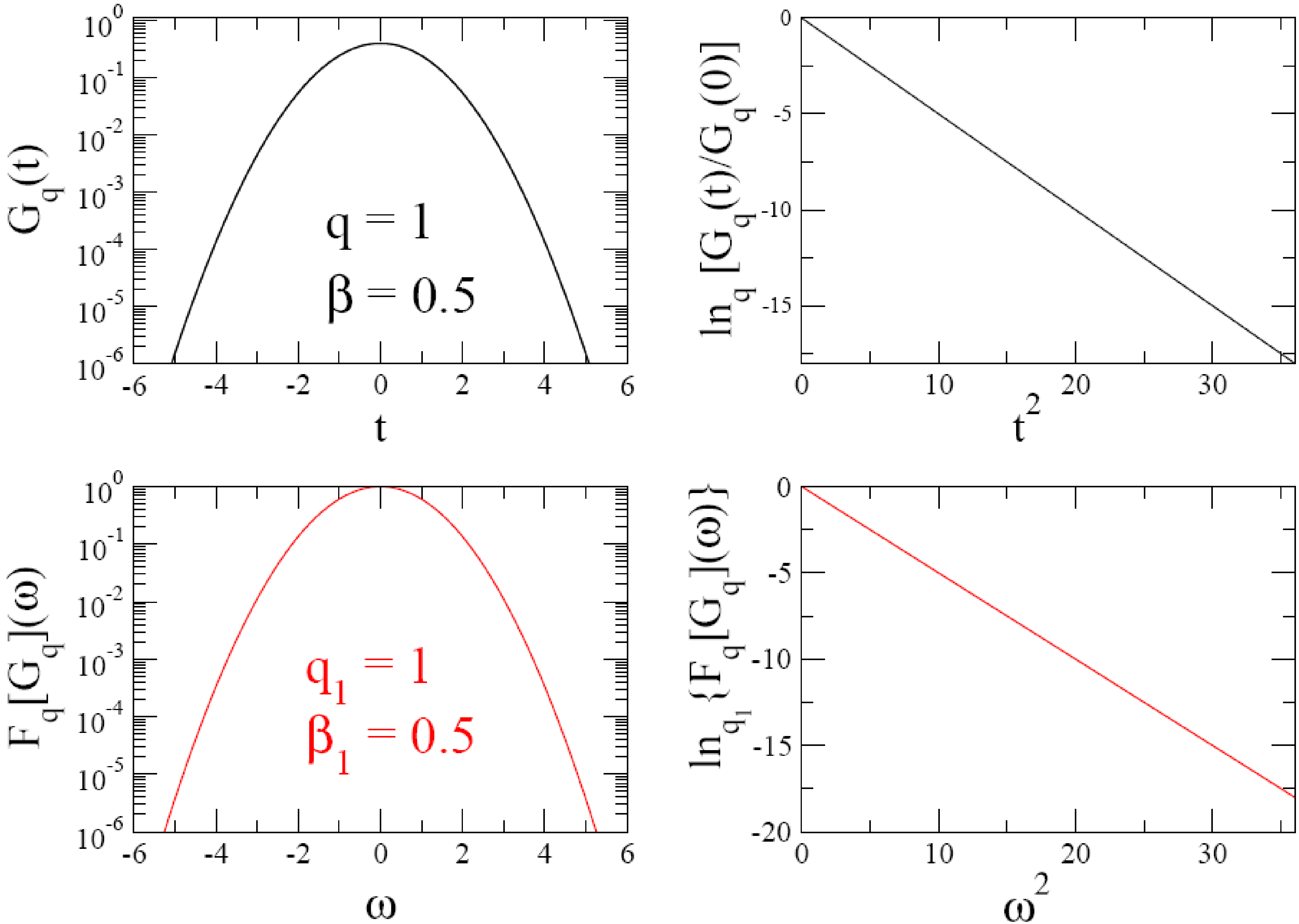
A. Robledo, Physica D **193**, 153 (2004)

$$q-FourierTransform\left[\frac{\sqrt{\beta}}{C_q}e_q^{-\beta\, t^2}\right]=e_{q_1}^{-\beta_1\,\omega^2}$$

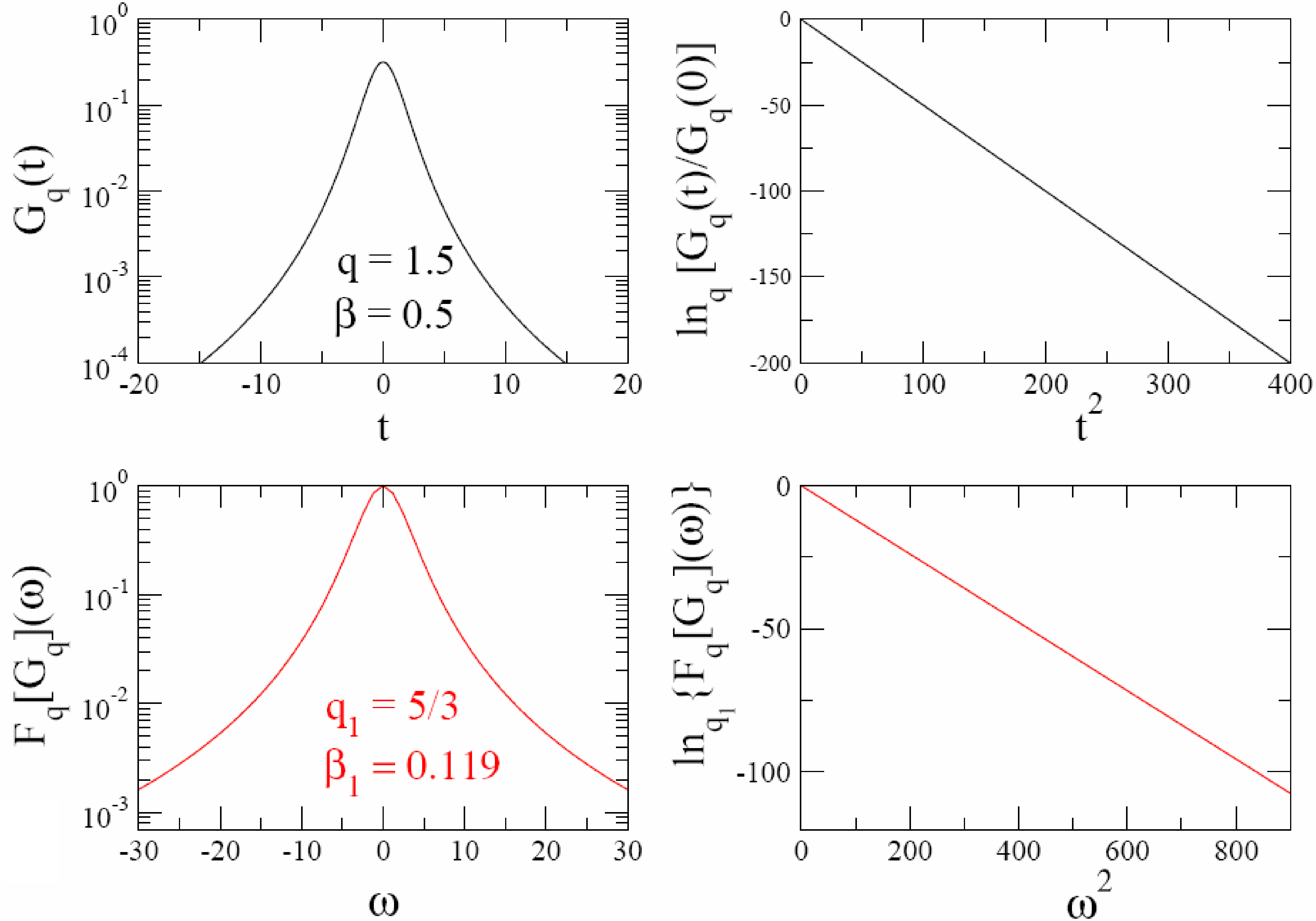
where $$q_1=\frac{1+q}{3-q}$$

and $$\beta_1=\frac{3-q}{8\beta^{2-q}C_q^{2(1-q)}}$$

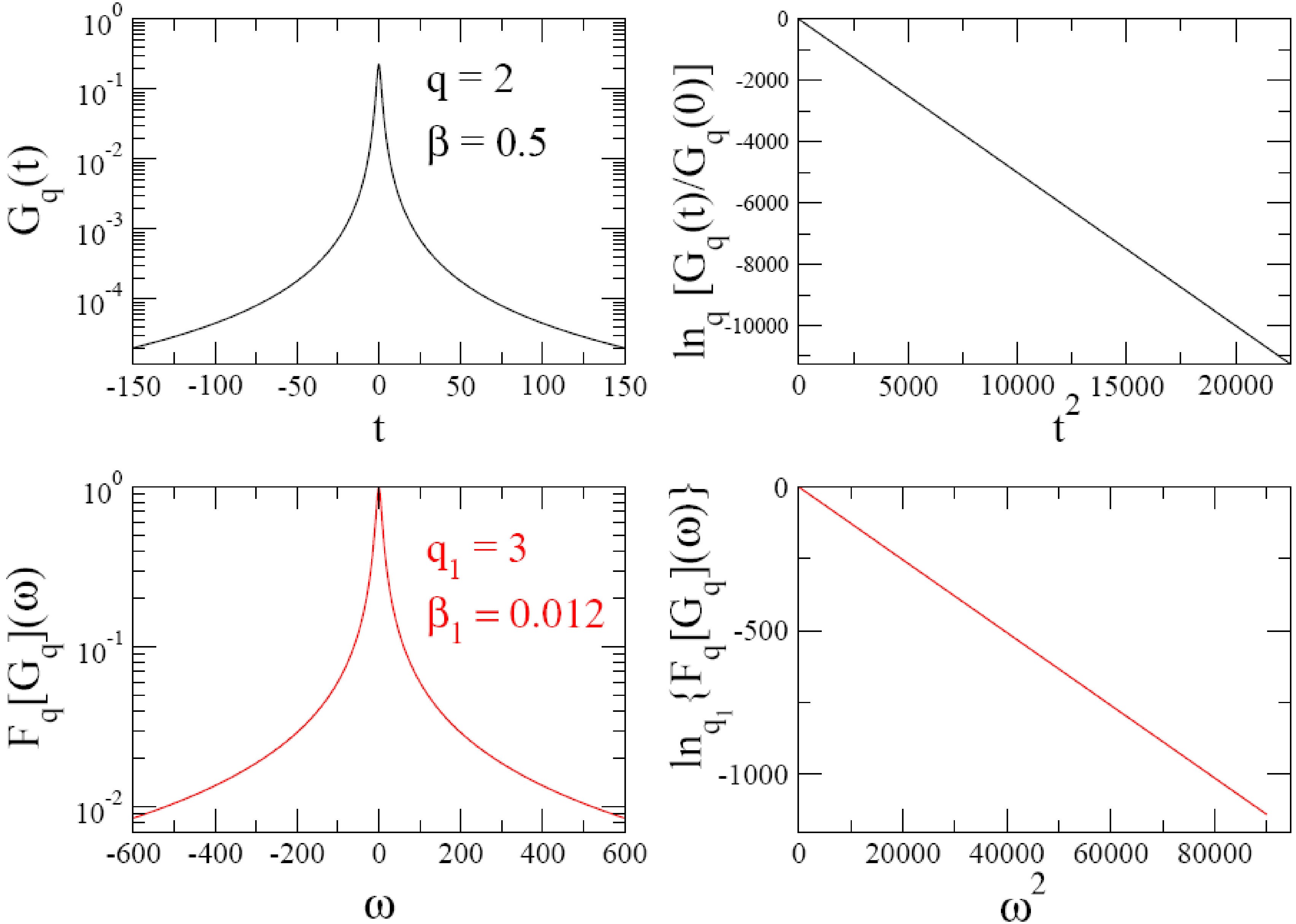
with $$C_q=\begin{cases}\dfrac{2\sqrt{\pi}\,\Gamma\left(\dfrac{1}{q-1}\right)}{(3-q)\sqrt{(1-q)}\,\Gamma\left(\dfrac{3-q}{2(1-q)}\right)} & \text{if } q<1\\ \sqrt{\pi} & \text{if } q=1\\ \dfrac{\sqrt{\pi}\,\Gamma\left(\dfrac{3-q}{2(q-1)}\right)}{\sqrt{q-1}\,\Gamma\left(\dfrac{1}{q-1}\right)} & \text{if } 1<q<3\end{cases}$$



$G_q(t)$
$q = 1$
$\beta = 0.5$
t
$\ln_q [G_q(t)/G_q(0)]$
t^2
$F_q[G_q](\omega)$
$q_1 = 1$
$\beta_1 = 0.5$
ω
$\ln_{q_1}\{F_q[G_q](\omega)\}$
ω^2



$G_q(t)$
t
$q = 1.5$
$\beta = 0.5$
$\ln_q[G_q(t)/G_q(0)]$
t^2
$F_q[G_q](\omega)$
ω
$q_1 = 5/3$
$\beta_1 = 0.119$
$\ln_{q_1}\{F_q[G_q](\omega)\}$
ω^2



q = 2
β = 0.5
G_q(t)
t
ln_q [G_q(t)/G_q(0)]
t^2
q_1 = 3
β_1 = 0.012
F_q[G_q](ω)
ω
ln_{q_1}{F_q[G_q](ω)}
ω^2

A random variable X is said to have a (q,α)*-stable distribution* $L_{q,\alpha}(x)$ *if its q-Fourier transform has the form* $a\ e_q^{-b\,|\xi|^\alpha}$ $(a>0,\ b>0,\ 0<\alpha\le 2)$

i.e., if

$$F_q[L_{q,\alpha}](\xi) \equiv \int_{-\infty}^{\infty} e_q^{ix\xi} \otimes_q L_{q,\alpha}(x)\,dx = \int_{-\infty}^{\infty} e_q^{\frac{ix\xi}{[L_{q,\alpha}(x)]^{1-q}}} L_{q,\alpha}(x)\,dx = a\ e_q^{-b\,|\xi|^\alpha}$$

$L_{1,2}(x) \equiv G(x)$ (*Gaussian*)

$L_{1,\alpha}(x) \equiv L_\alpha(x)$ (α *– stable Levy distribution*)

$L_{q,2}(x) \equiv G_q(x)$ (q *– Gaussian*)

S. Umarov, C. T., M. Gell-Mann and S. Steinberg (2006)

cond-mat/0606038

cond-mat/0606040

CENTRAL LIMIT THEOREMS: $N^{1/[\alpha(2-q)]}$ - *SCALED ATTRACTOR* $\mathbb{F}(x)$ *WHEN SUMMING* $N \to \infty$

q - *CORRELATED IDENTICAL RANDOM VARIABLES WITH SYMMETRIC DISTRIBUTION* $f(x)$

	$q=1$ [*independent*]	$q \neq 1$ (*i.e.*, $Q \equiv 2q-1 \neq 1$) [*globally correlated*]
$\sigma_Q < \infty$ $(\alpha = 2)$	$\mathbb{F}(x) = $ *Gaussian* $G(x)$, *with same* σ_1 *of* $f(x)$ Classic CLT	$\mathbb{F}(x) = G_{\frac{3q-1}{q+1}}(x) \equiv \frac{3q-1}{q+1}$ - *Gaussian*, *with same* σ_Q $\left[\equiv \int dx\, x^2 [f(x)]^Q / \int dx\, [f(x)]^Q\right]$ *of* $f(x)$ $G_{\frac{3q-1}{q+1}}(x) \begin{cases} \simeq G(x) & \text{if } \lvert x \rvert \ll x_c(q,2) \\ \sim f(x) \sim C_q / \lvert x \rvert^{(q+1)/(q-1)} & \text{if } \lvert x \rvert \gg x_c(q,2) \end{cases}$ *with* $\lim_{q \to 1} x_c(q,2) = \infty$ S. Umarov, C. T. and S. Steinberg (2006) [cond-mat/0603593]
$\sigma_Q \to \infty$ $(0 < \alpha < 2)$	$\mathbb{F}(x) = $ *Levy distribution* $L_\alpha(x)$, *with same* $\lvert x \rvert \to \infty$ *asymptotic behavior* $L_\alpha(x) \begin{cases} \simeq G(x) \\ \quad \text{if } \lvert x \rvert \ll x_c(1,\alpha) \\ \sim f(x) \sim C_\alpha / \lvert x \rvert^{1+\alpha} \\ \quad \text{if } \lvert x \rvert \gg x_c(1,\alpha) \end{cases}$ *with* $\lim_{\alpha \to 2} x_c(1,\alpha) = \infty$ Levy-Gnedenko CLT	$\mathbb{F}(x) = L_{\frac{2\alpha q - \alpha + 3}{\alpha + 1}, \alpha}$ *stable distribution*, *with* $L_{\frac{2\alpha q - \alpha + 3}{\alpha + 1}, \alpha} \sim f(x) \sim C^{(L)}_{q,\alpha} / \lvert x \rvert^{(1+\alpha)/(1+\alpha q - \alpha)}$ *or* $\mathbb{F}(x) = L_{\frac{\alpha q + q - 1}{\alpha + q - 1}, \alpha}$ *stable distribution*, *with* $L_{\frac{\alpha q + q - 1}{\alpha + q - 1}, \alpha} \sim f(x) \sim C^{(*)}_{q,\alpha} / \lvert x \rvert^{2(\alpha + q - 1)/\alpha(q-1)}$ S. Umarov, C. T., M. Gell-Mann and S. Steinberg (2006) [cond-mat/0606038] and [cond-mat/0606040]

BOLTZMANN-GIBBS STATISTICAL MECHANICS

(Maxwell 1860, Boltzmann 1872, Gibbs ≤ 1902)

Entropy $$S_{BG} = -k\sum_{i=1}^{W} p_i \ln p_i$$

Internal energy $$U_{BG} = \sum_{i=1}^{W} p_i E_i$$

Equilibrium distribution $$p_i = e^{-\beta E_i} / Z_{BG} \qquad \left(Z_{BG} \equiv \sum_{j=1}^{W} e^{-\beta E_j} \right)$$

Paradigmatic differential equation $$\left.\begin{array}{l} \frac{dy}{dx} = a\,y \\ y(0) = 1 \end{array}\right\} \Rightarrow \qquad y = e^{ax}$$

	x	a	$y(x)$
Equilibrium distribution	E_i	$-\beta$	$Z\,p(E_i)$
Sensitivity to initial conditions	t	λ	$\xi \equiv \lim_{\Delta x(0)\to 0} \frac{\Delta x(t)}{\Delta x(0)} = e^{\lambda t}$
Typical relaxation of observable O	t	$-1/\tau$	$\Omega \equiv \frac{O(t) - O(\infty)}{O(0) - O(\infty)} = e^{-t/\tau}$

S_{BG} → extensive, concave, Lesche-stable, finite entropy production

NONEXTENSIVE STATISTICAL MECHANICS

(C. T. 1988, E.M.F. Curado and C. T. 1991, C. T., R.S. Mendes and A.R. Plastino 1998)

Entropy

$$S_q = k\left(1 - \sum_{i=1}^{W} p_i^q\right)/(q-1)$$

Internal energy

$$U_q = \sum_{i=1}^{W} p_i^q E_i \Big/ \sum_{j=1}^{W} p_j^q$$

Stationary state distribution

$$p_i = e_q^{-\beta_q (E_i - U_q)} / Z_q \qquad \left(Z_q \equiv \sum_{j=1}^{W} e_q^{-\beta_q (E_j - U_q)} \right)$$

Paradigmatic differential equation

$$\left.\begin{aligned} \frac{dy}{dx} &= a\, y^q \\ y(0) &= 1 \end{aligned}\right\} \Rightarrow \qquad y = e_q^{ax} \equiv \left[1 + (1-q)\,a x\right]^{\frac{1}{1-q}}$$

	x	a	$y(x)$
Stationary state distribution	E_i	$-\beta_{q_{stat}}$	$Z_{q_{stat}}\, p(E_i)$ (typically $q_{stat} \geq 1$)
Sensitivity to initial conditions	t	$\lambda_{q_{sen}}$	$\xi = e_{q_{sen}}^{\lambda_{q_{sen}} t}$ (typically $q_{sen} \leq 1$)
Typical relaxation of observable O	t	$-1/\tau_{q_{rel}}$	$\Omega = e_{q_{rel}}^{-t/\tau_{q_{rel}}}$ (typically $q_{rel} \geq 1$)

S_q → extensive, concave, Lesche-stable, finite entropy production

C. T., Physica A **340**,1 (2004)

Prediction of the q - triplet: C. T., Physica A **340**,1 (2004)

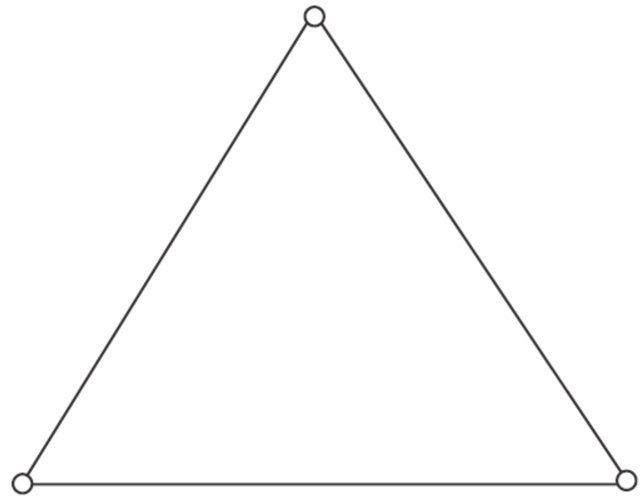
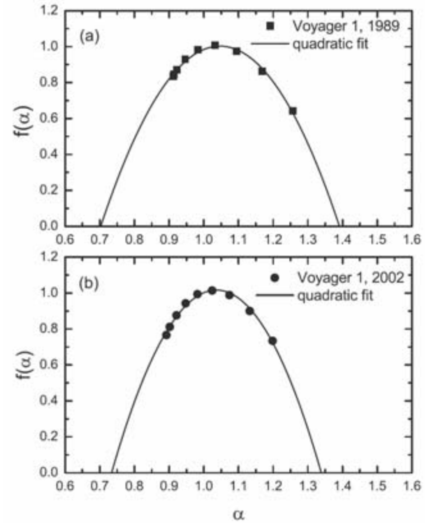


Fig. 2. The triangle of the basic values of q, namely those associated with sensitivity to the initial conditions, relaxation and stationary state. For the most relevant situations we expect $q_{sen} \leqslant 1$, $q_{rel} \geqslant 1$ and $q_{stat} \geqslant 1$. These indices are presumably inter-related since they all descend from the particular dynamical exploration that the system does of its full phase space. For example, for long-range Hamiltonian systems characterized by the decay exponent α and the dimension d, it could be that q_{stat} decreases from a value above unity (e.g., 2 or $\frac{3}{2}$) to unity when α/d increases from zero to unity. For such systems one expects relations like the (particularly simple) $q_{stat} = q_{rel} = 2 - q_{sen}$ or similar ones. In any case, it is clear that, for $\alpha/d > 1$ (i.e., when BG statistics is known to be the correct one), one has $q_{stat} = q_{rel} = q_{sen} = 1$. All the weakly chaotic systems focused on here are expected to have well defined values for q_{sen} and q_{rel}, but only those associated with a Hamiltonian are expected to *also* have a well defined value for q_{stat}.

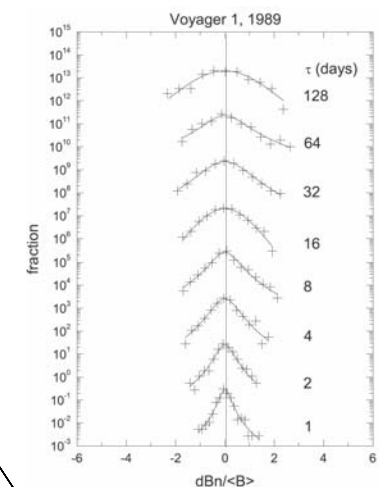
SOLAR WIND: Magnetic Field Strength

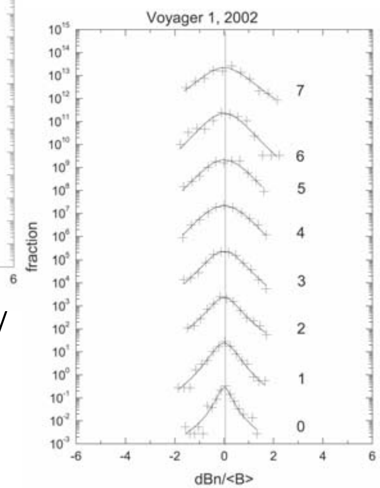
L.F. Burlaga and A. F.-Vinas (2005) / NASA Goddard Space Flight Center; Physica A **356**, 375 (2005)

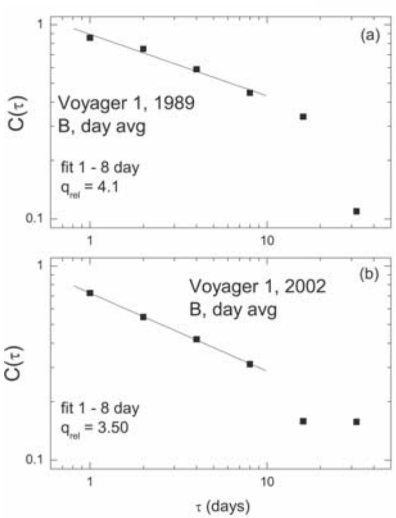
[Data: Voyager 1 spacecraft (1989 and 2002); 40 and 85 AU; daily averages]



$$q_{sen} = -0.6 \pm 0.2$$







$$q_{rel} = 3.8 \pm 0.3$$

$$q_{stat} = 1.75 \pm 0.06$$

Playing with additive duality $(q \to 2-q)$

and with multiplicative duality $(q \to 1/q)$

(*and using numerical results related to the q–generalized central limit theorem*)

we conjecture

$$q_{rel} + \frac{1}{q_{sen}} = 2 \qquad \text{and} \qquad q_{stat} + \frac{1}{q_{rel}} = 2$$

hence $$1 - q_{sen} = \frac{1-q_{stat}}{3-2\,q_{stat}}$$

hence only one independent!

Burlaga and Vinas (NASA) most precise value of the q–triplet is

$$q_{stat} = 1.75 = 7/4$$

hence $q_{sen} = -\,0.5 = -\,1/2$ (*consistent with* $q_{sen} = -\,0.6 \pm 0.2$!)

and $q_{rel} = 4$ (*consistent with* $q_{rel} = 3.8 \pm 0.3$!)

C.T., M. Gell-Mann and Y. Sato

Proc Natl Acad Sc USA **102**, 15377 (2005)

Connections with asymptotically scale – free networks

GEOGRAPHIC PREFERENTIAL ATTACHMENT GROWING NETWORK:

D.J.B. Soares, C. T. , A.M. Mariz and L.R. Silva, Europhys Lett **70**, 70 (2005)

(1) Locate site $i=1$ at the origin of say a plane

(2) Then locate the next site with

$$P_G \propto 1/r^{2+\alpha_G} \quad (\alpha_G \geq 0)$$

$(r \equiv \textit{distance to the baricenter of the pre-existing cluster})$

(3) Then link it to only one of the previous sites using

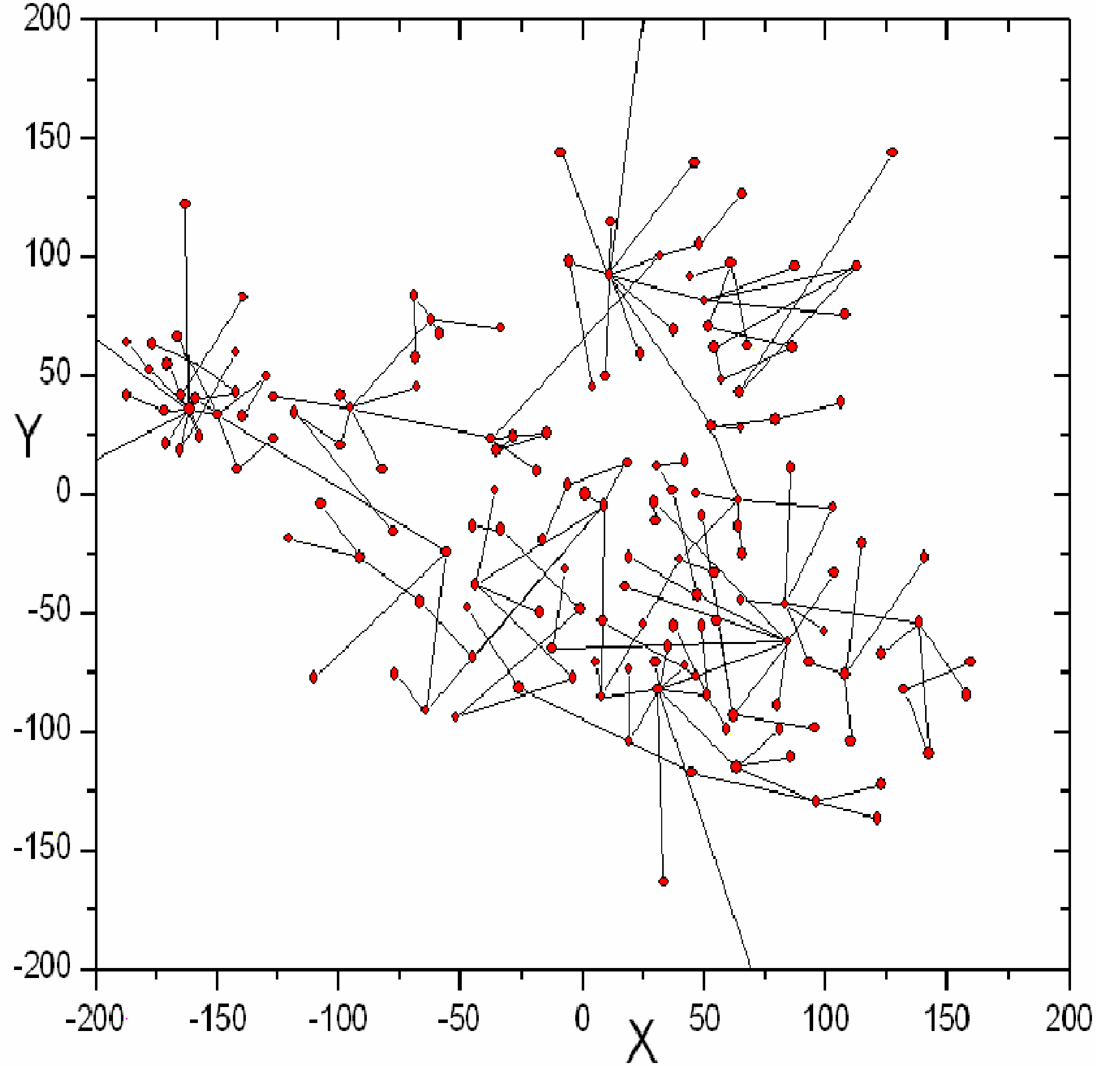
$$p_A \propto k_i / r_i^{\alpha_A} \quad (\alpha_A \geq 0)$$

$(k_i \equiv \textit{links already attached to site } i)$

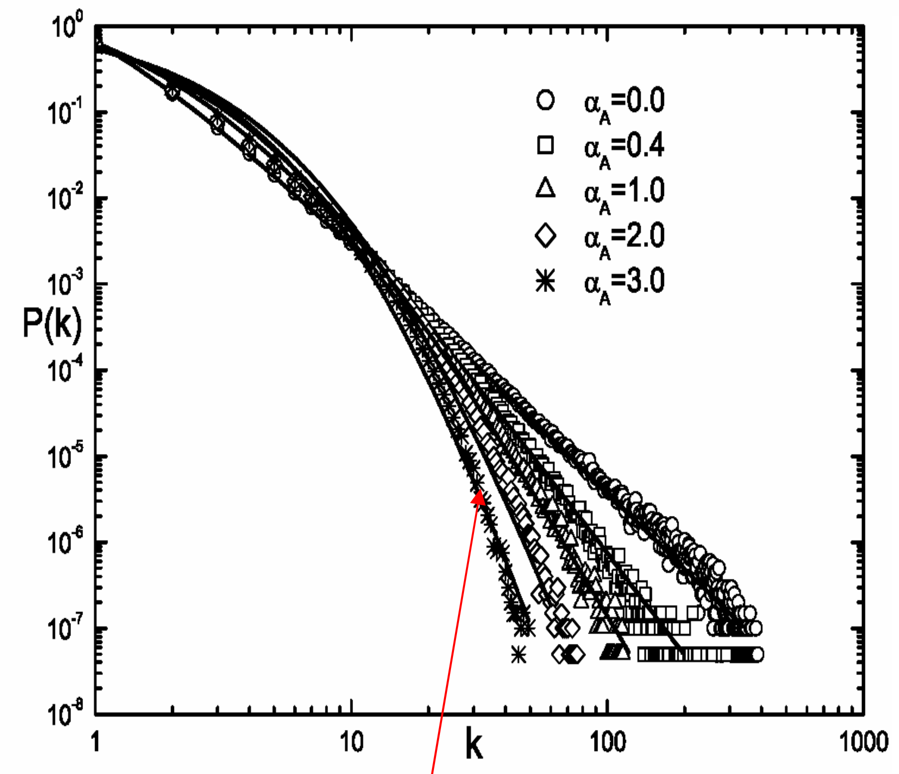
$(r_i \equiv \textit{distance to site } i)$

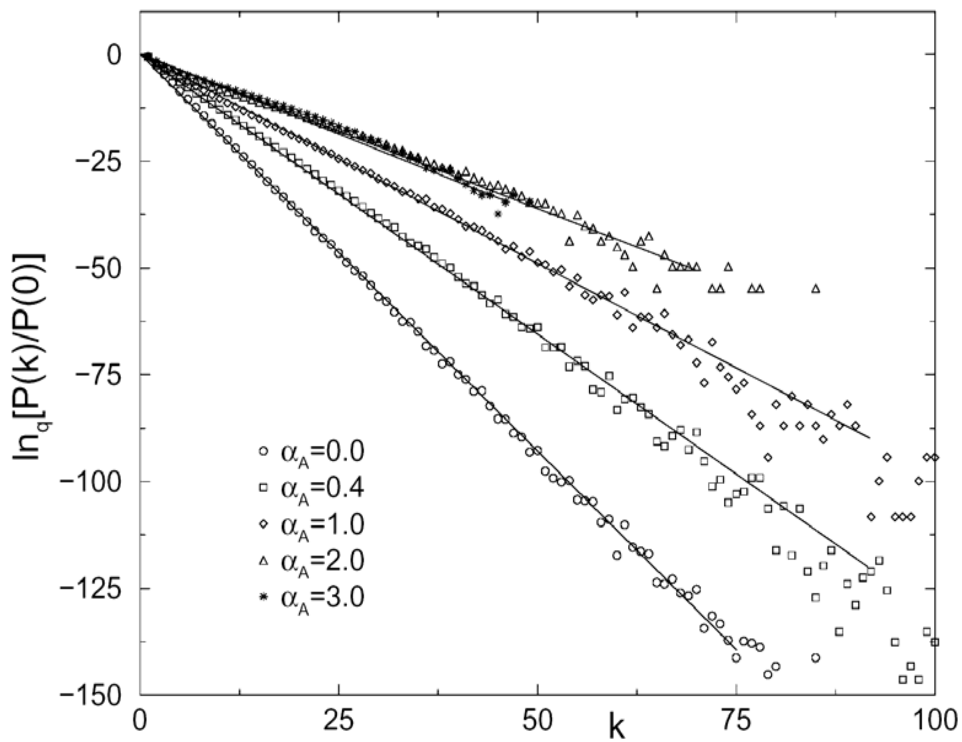
4) Repeat

$$(\alpha_G = 1; \alpha_A = 1; N = 250)$$



D.J.B. Soares, C. T. , A.M. Mariz and L.R. Silva
Europhys Lett **70**, 70 (2005)





$$P(k)/P(0) = e_q^{-k/\kappa}$$

$$\equiv 1/[1+(q-1)k/\kappa]^{1/(q-1)}$$

D.J.B. Soares, C. T. , A.M. Mariz and L.R. Silva
Europhys Lett **70**, 70 (2005)

PREDICTION:

The solution of

$$\frac{\partial\, p(x,t)}{\partial t} = D\, \frac{\partial^2 [p(x,t)]^{2-q}}{\partial x^2} \qquad [p(x,0) = \delta(0)] \qquad (q < 3)$$

is given by

$$p(x,t) \propto \left[1 + (1-q)\, x^2 \,/\, (\Gamma t)^{2/(3-q)}\right]^{1/(1-q)} \equiv e_q^{-x^2 /\, (\Gamma t)^{2/(3-q)}} \qquad (\Gamma \propto D)$$

hence

x^2 *scales like* t^{γ} $\left(e.g., \left\langle x^2 \right\rangle \propto t^{\gamma}\right)$

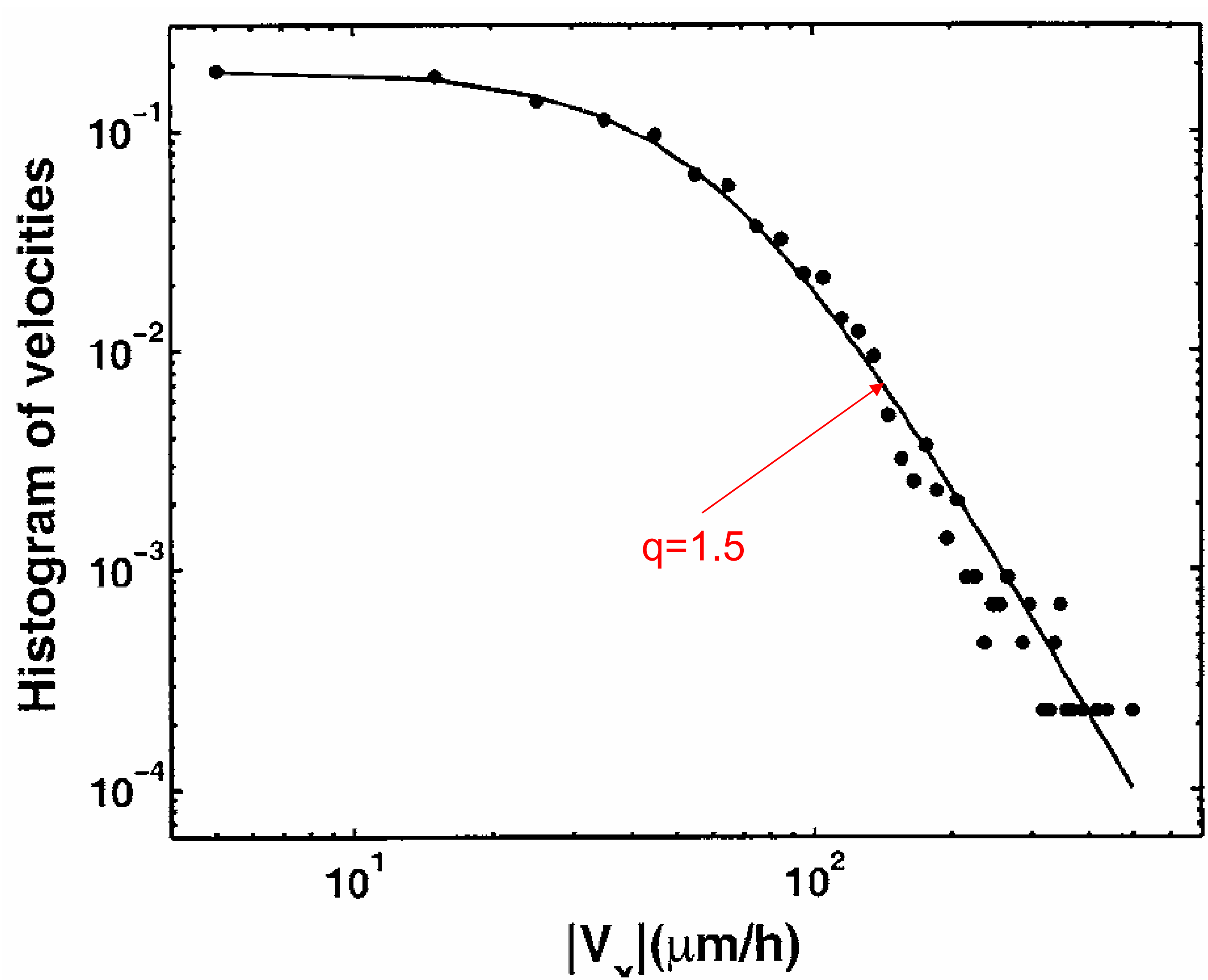
with

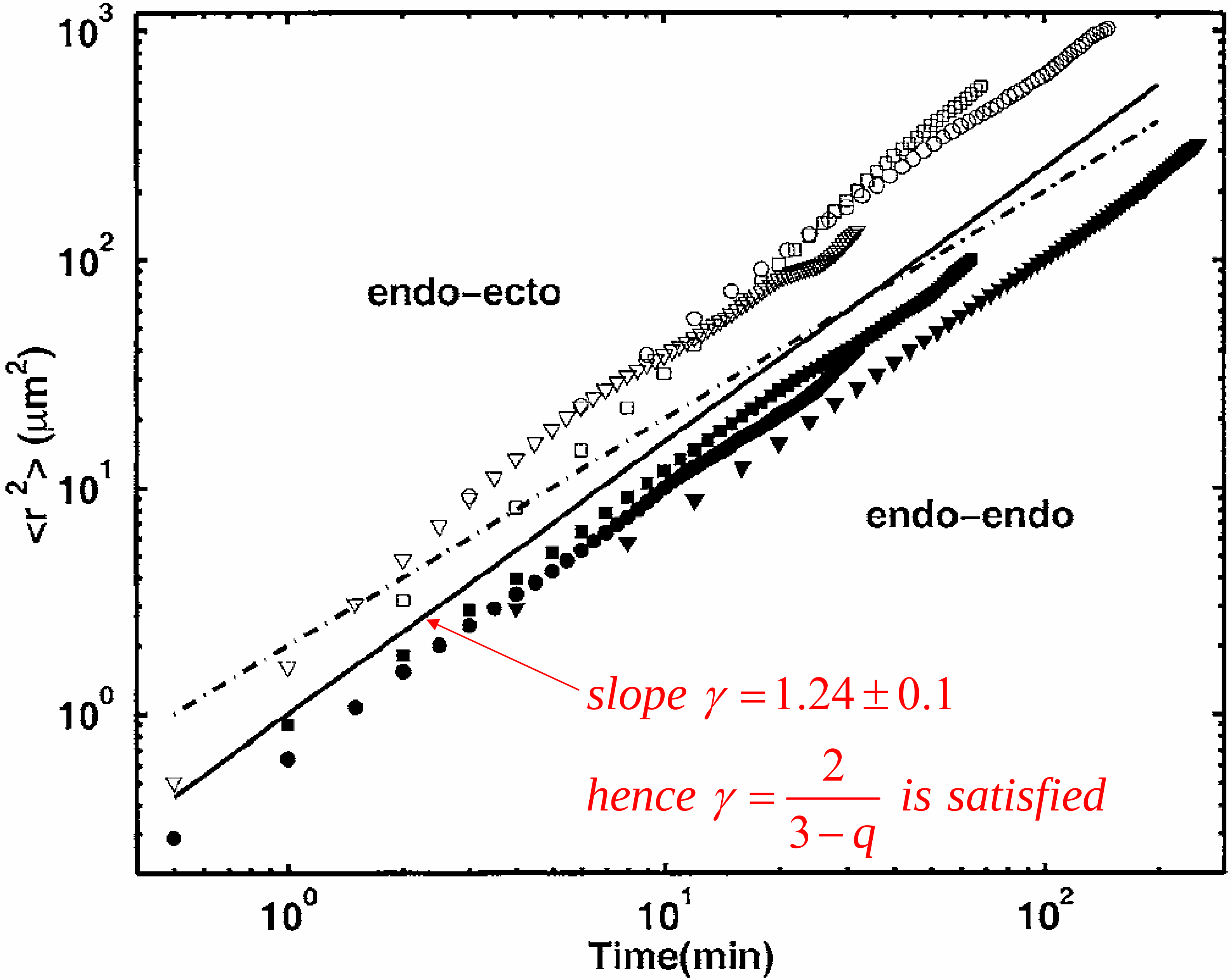
$$\gamma = \frac{2}{3-q}$$

C.T. and D.J. Bukman, Phys Rev E **54**, R2197 (1996)

Hydra viridissima:

A. Upadhyaya, J.-P. Rieu, J.A. Glazier and Y. Sawada
Physica A **293**, 549 (2001)

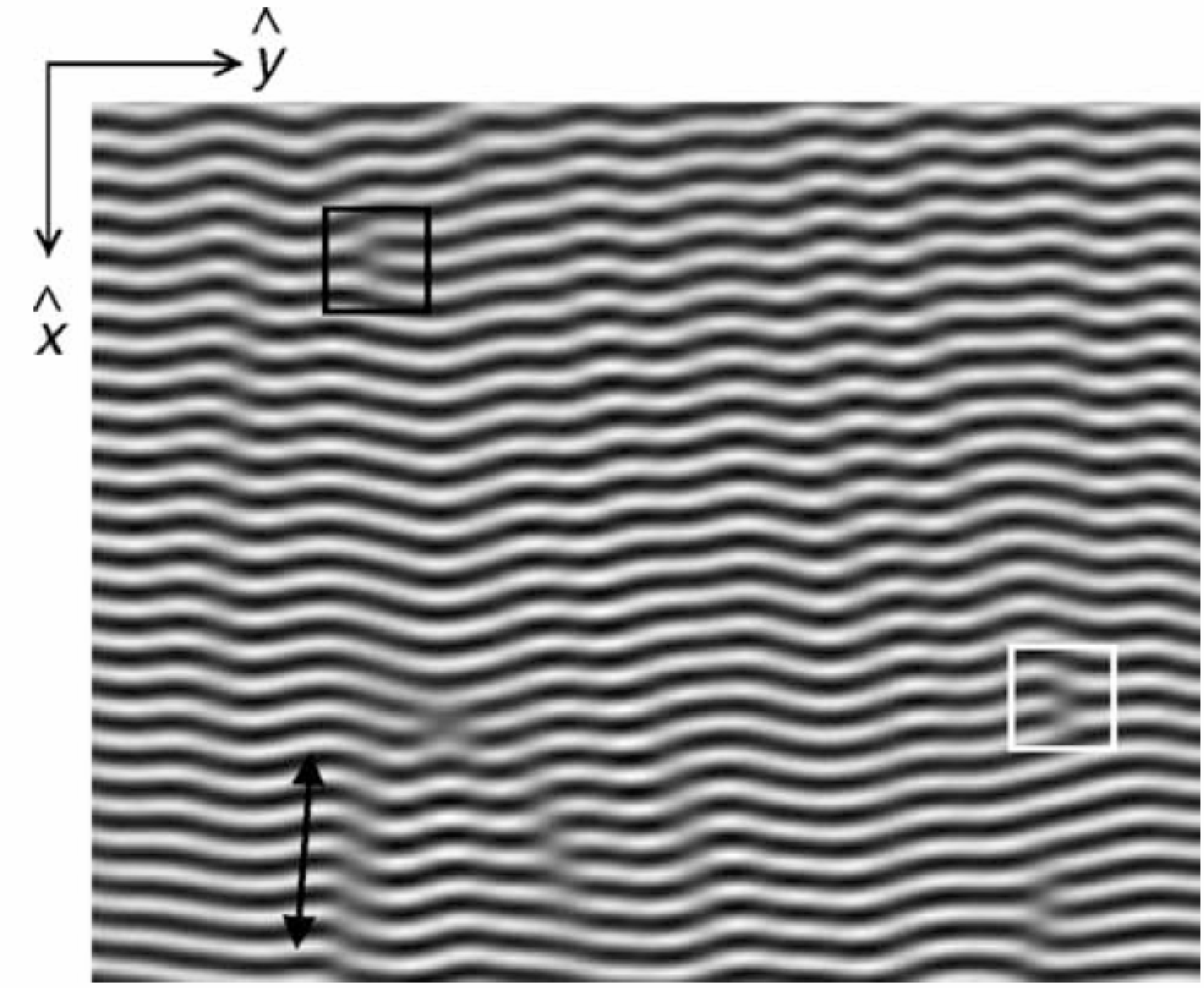


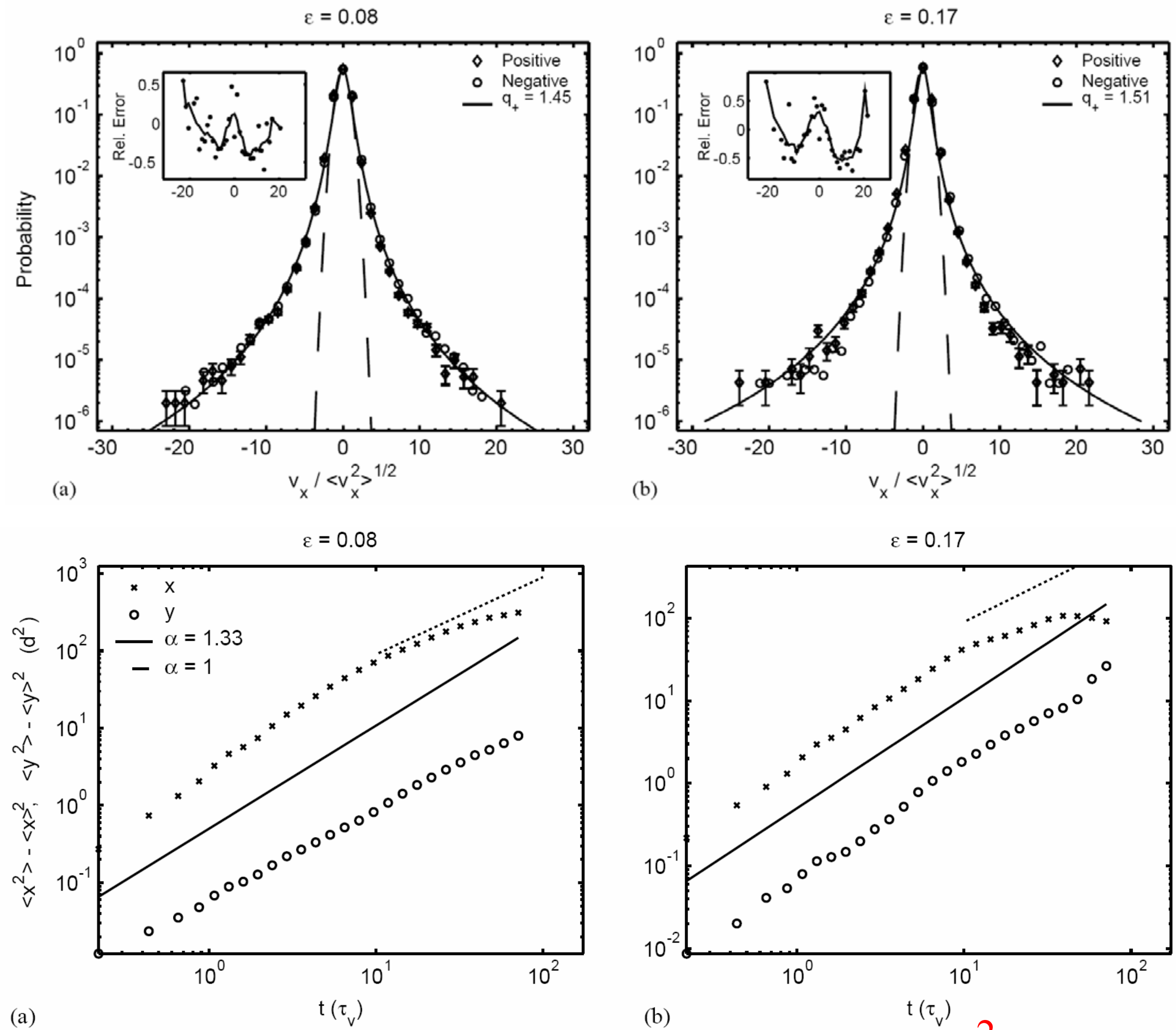


endo-ecto
endo-endo
$\langle r^2 \rangle$ (μm^2)
Time(min)
10^0
10^1
10^2
10^3
slope $\gamma = 1.24 \pm 0.1$
hence $\gamma = \frac{2}{3-q}$ is satisfied

Defect turbulence:

K.E. Daniels, C. Beck and E. Bodenschatz, Physica D **193**, 208 (2004)

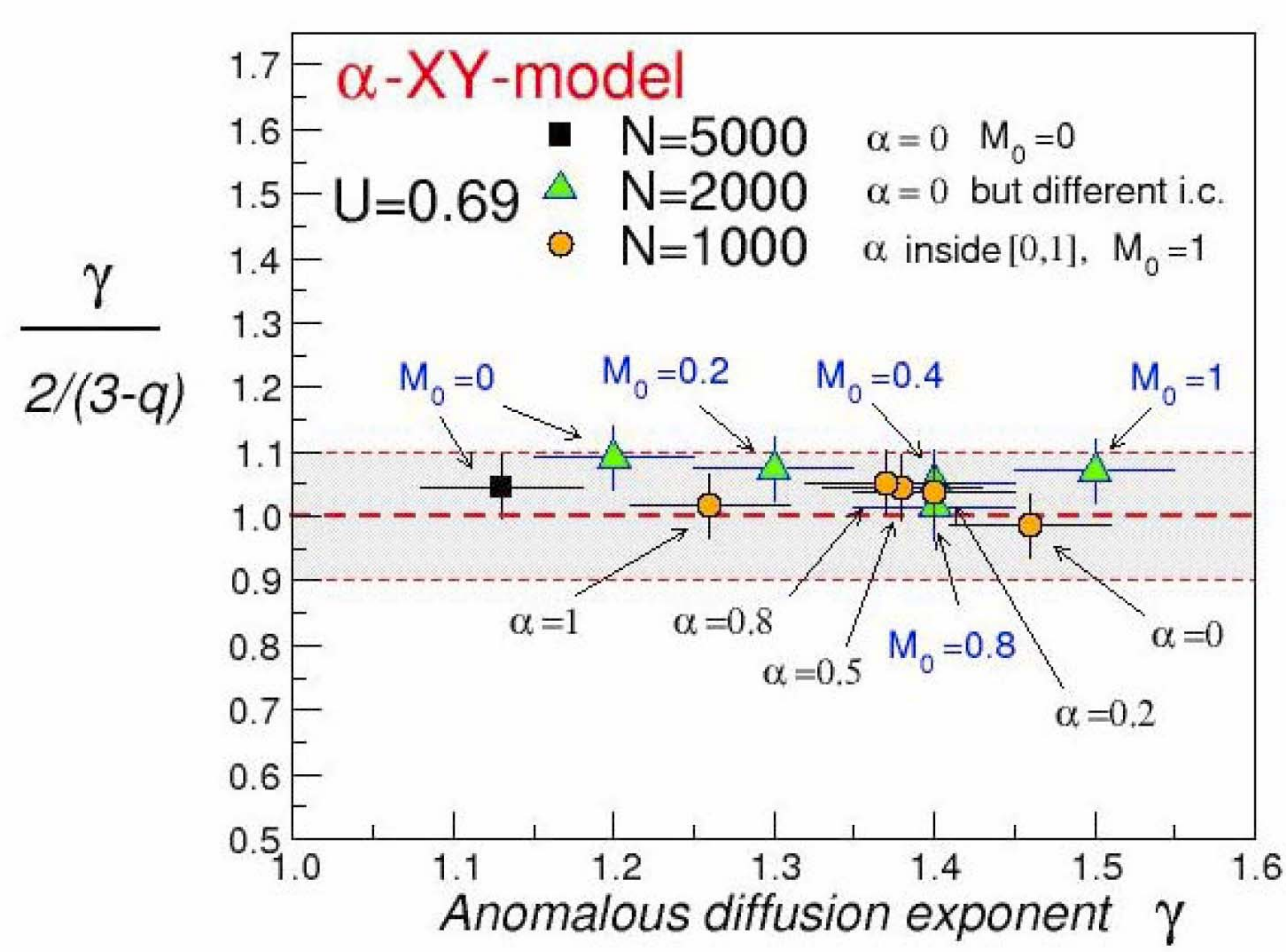




$q \approx 1.5$ *and* $\gamma \approx 4/3$ *are consistent with* $\gamma = \dfrac{2}{3-q}$

K.E. Daniels, C. Beck and E. Bodenschatz, Physica D **193**, 208 (2004)

XY FERROMAGNET WITH LONG-RANGE INTERACTIONS:



A. Rapisarda and A. Pluchino, Europhys News **36**, 202 (2005)
(European Physical Society)

COLD ATOMS IN DISSIPATIVE OPTICAL LATTICES:

Theoretical predictions by E. Lutz, Phys Rev A **67**, 051402(R) (2003):

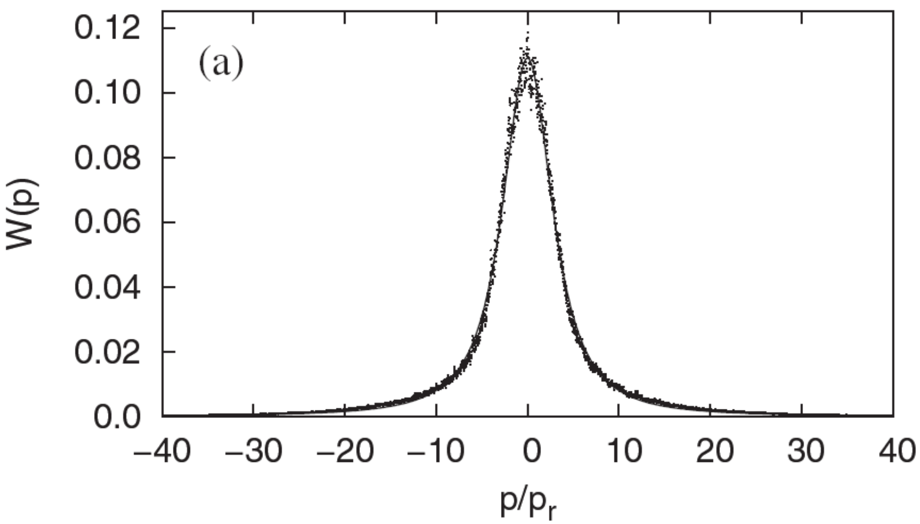
(i) The distribution of atomic velocities is a *q*-Gaussian;

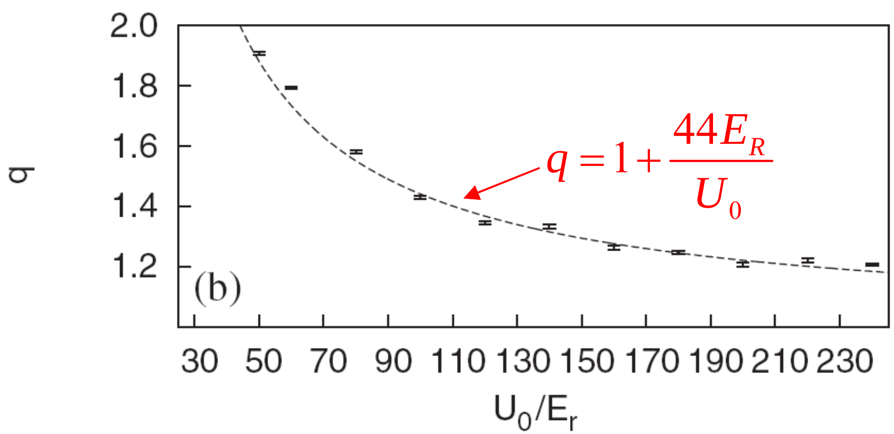
(ii) $q = 1 + \dfrac{44 E_R}{U_0}$ where $E_R \equiv$ recoil energy

$U_0 \equiv$ potential depth

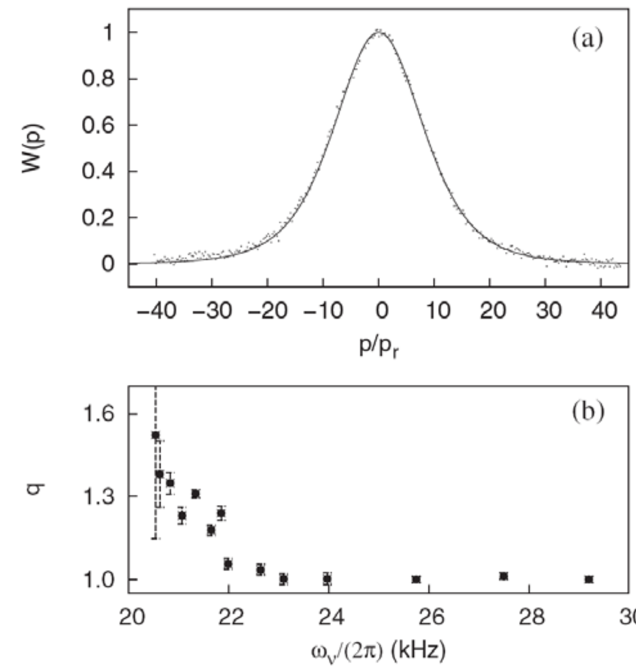
Experimental and computational verifications

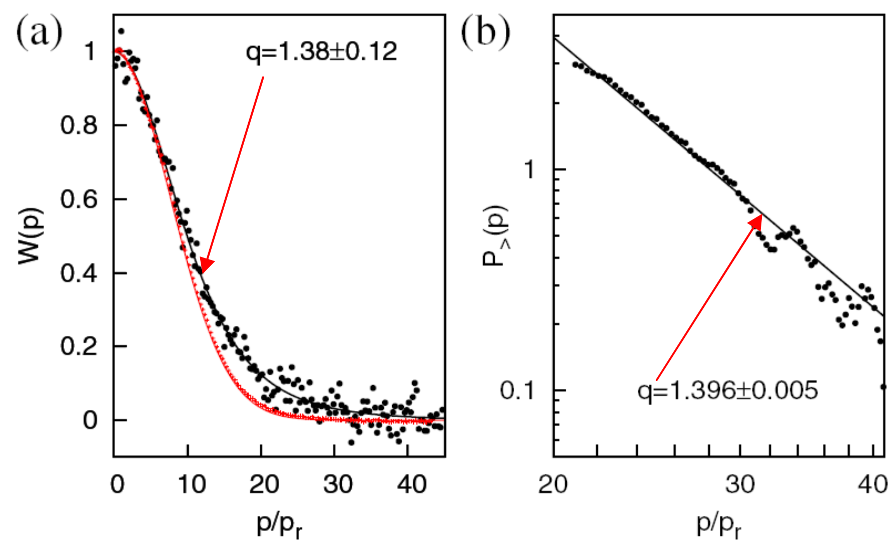
by P. Douglas, S. Bergamini and F. Renzoni, Phys Rev Lett **96**, 110601 (2006)





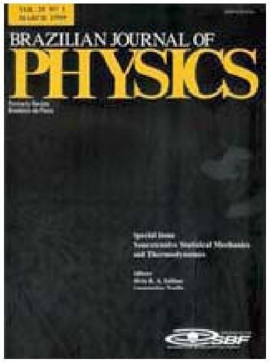
(Computational verification: quantum Monte Carlo simulations)



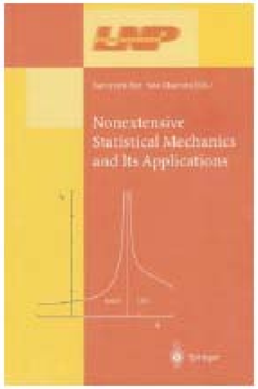


(Experimental verification)

NONEXTENSIVE STATISTICAL MECHANICS AND THERMODYNAMICS

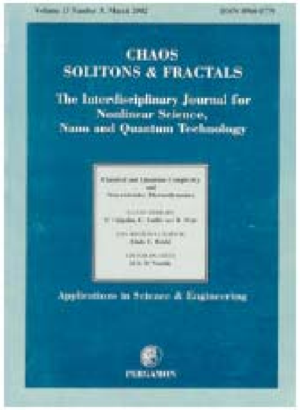


Nonextensive Statistical Mechanics and Thermo-dynamics
SRA Salinas and C Tsallis, eds
Brazilian Journal of Physics
29, Number 1 (**1999**)

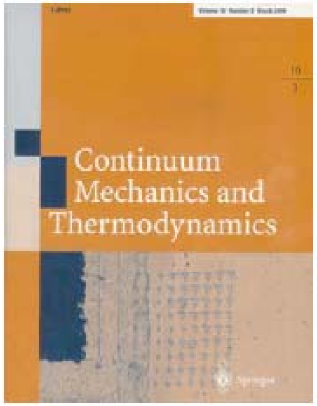


Nonextensive Statistical Mechanics and Its Applications
S Abe and Y Okamoto, eds
Lectures Notes in Physics
(Springer, Berlin, **2001**)

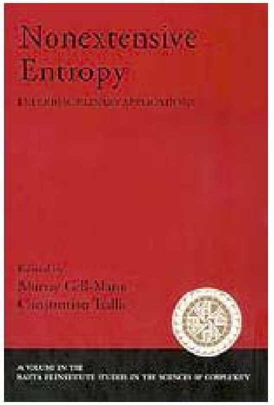
Non Extensive Thermodynamics and Physical Applications
G Kaniadakis, M Lissia and A Rapisarda, eds
Physica A **305**, Issue 1/2
(**2002**)



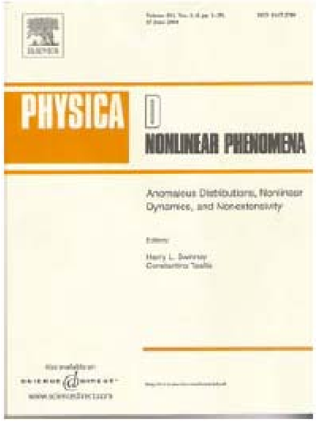
Classical and Quantum Complexity and Nonextensive Thermodynamics
P Grigolini, C Tsallis
and BJ West, eds
Chaos, Solitons and Fractals
13, Issue 3 (**2002**)



Nonadditive Entropy and Nonextensive Statistical Mechanics
M. Sugiyama, ed
Continuum Mechanics and Thermodynamics **16** (Springer, Heidelberg, **2004**)



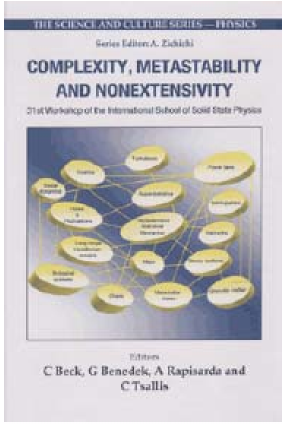
Nonextensive Entropy - Interdisciplinary Applications
M Gell-Mann and C Tsallis, eds
(Oxford University Press, New York, **2004**)



Anomalous Distributions, Nonlinear Dynamics, and Nonextensivity
HL Swinney and C Tsallis, eds
Physica D **193**, Issue 1-4 (**2004**)

News and Expectations in Thermostatistics
G Kaniadakis and M Lissia, eds
Physica A **340**, Issue 1/3
(**2004**)

Trends and Perspectives in Extensive and Non-Extensive Statistical Mechanics
H Herrmann, M Barbosa and E Curado, eds
Physica A **344**, Issue 3/4
(**2004**)



Complexity, Metastability and Nonextensivity
C Beck, G Benedek, A Rapisarda and C Tsallis, eds
(World Scientific, Singapore, **2005**)

Nonextensive Statistical Mechanics: New Trends, New Perspectives JP Boon and C Tsallis, eds Europhysics News (European Physical Society, **2005**)

Fundamental Problems of Modern Statistical Mechanics
G Kaniadakis, A Carbone and M. Lissia, eds Physica A **365**, Issue 1 (**2006**)

$than_q$