Some uniform ergodic theorems via skew-products

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(joint work with S. Santos)

Birkhoff's Ergodic Theorem

Suppose: • $(X, \mathcal{B}, \mu)$ = probability space

• $T: X \to X$ ergodic measure-preserving map

• $f \in \mathcal{L}^1(X)$

Then: $$\frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) \to \int f \, d\mu \quad \text{a.e.}$$

"Topological" Ergodic Theorem

Suppose: • $T: X \to X$ continuous transformation of a compact metric space X

• μ = ergodic Borel probability measure

Then:

$\exists \; G_\mu = \{\text{generic points}\} \subset X, \quad \mu(G_\mu) = 1,$

s.t. $$\frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) \to \int f \, d\mu$$

$$\forall f \in C(X, \mathbb{R}), \quad \forall x \in G_\mu.$$

untner Ergodic Theorem

- $T: X \to X$ ergodic m.p.t. of a probability space $(X, \mathcal{B}, \mu)$.
- $f \in L^2(X)$

$z \in K$ = unit circle in $\mathbb{C}$

a projection operator $P_z : L^2 \to L^2$

$$\frac{1}{n} \sum_{j=0}^{n-1} z^j f(T^j x) \longrightarrow P_z f(x) \quad \text{a.e. } x. \qquad (*)$$

$z = 1$ this is Birkhoff's Ergodic Thm.

et of full measure is independent of z

$\in L^2$ $\exists X_f$, $\mu(X_f) = 1$, s.t.

$\forall z \in K$, $(*)$ converges

of T

es a unitary operator

$\mapsto \omega \circ T : L^2 \to L^2$

eigenvalue for T if $\exists\, 0 \neq \omega \in L^2$

$\omega \circ T = z\,\omega$

ergodic if 1 is a simple eigenvalue

$T = \omega \implies \omega$ constant).

-W theorem

, if z is not an eigenvalue

thogonal projection onto eigenspace

Sketch of link between eigenvalues & W-W.

- Fix $z \in K \setminus \{1\}$. Define the transformation

$$T_z : X \times K \longrightarrow X \times K$$
$$(x, y) \longmapsto (Tx, yz)$$

(preserves $\mu \times$ Lebesgue)

- T_z is ergodic $\iff$ z not an eigenvalue for T

⌈ Let $F(x,y) = yf(x) : X \times K \to \mathbb{C}$, $f : X \to \mathbb{R}$.

Then

- $F \circ T_z = F \iff yzf(Tx) = yf(x)$
$\iff \bar{z}$ is an eigenvalue ⌋

- If z is not an eigenvalue then

$$\frac{1}{n}\sum_{j=0}^{n-1} z^j f(T^j x) = \frac{1}{n}\sum_{j=0}^{n-1} F T_z^j (x,1)$$

$$\xrightarrow{B.E.T} \int F (d\mu \times \text{Lebesgue})$$

$$= \underbrace{\int_K y \, dy}_{=0} \int_X f \, d\mu. \quad = 0$$

Skew-products

- $T : X \to X$ continuous transformation of a compact metric space X
- μ = ergodic Borel probability measure on X.
- G = compact Lie group, λ = Haar measure.
- $\phi : X \longrightarrow G$ continuous

Obtain a skew-product by defining

$$T_\phi : X \times G \longrightarrow X \times G$$
$$(x, y) \longmapsto (Tx, \; y\,\phi(x))$$

This preserves $\mu \times \lambda$, but is not nec. ergodic

Criteria for ergodicity (Keynes + Newton)

- $G = K$: T_ϕ not ergodic $\iff$ $\exists\, d \in \mathbb{Z} \setminus \{0\}$
  $\exists$ measurable $\omega : X \to K$
  st $\omega(Tx) = \phi(x)^d \, \omega(x)$

- G compact Lie : T_ϕ is not ergodic $\iff$ $\exists$ non-trivial unitary representation $R : G \to U(d)$
  $\exists$ measurable $\omega : X \to \mathbb{C}^d$
  st $\omega(Tx) = R(\phi x)\, \omega(x)$

Walters' Topological Wiener-Wintner Theorem

Suppose:
- T cts tx of cpct metric space X
- $\phi : X \to K$ cts
- $T_\phi : X \times K \to X \times K$ ergodic wrt $\mu \times$ Lebesgue

Then
- $\exists X_\mu \subset X$, $\mu(X_\mu) = 1$ st

$$(*) \quad \frac{1}{n} \sum_{j=0}^{n-1} \phi(x)\phi(Tx) \cdots \phi(T^{j-1}x) f(T^j x) \to 0$$

$$\forall f \in C(X, \mathbb{C}), \ \forall x \in X_\mu.$$

[When T_ϕ is not ergodic, $(*)$ converges to a limit $l(x)$ which satisfies $l(Tx) = \phi(x)\, l(x)$.]

Remarks

- The set X_μ is independent of ϕ
- The convergence in $(*)$ is uniform in ϕ, with ϕ chosen from a compact subset of $C(X; K)$ (if $\int f \, d\mu = 0$)

Application of Walters' W-W thm

Let $X = \{1, \dots, k\}^{\mathbb{Z}}$, $T: X \to X$ = shift

$$T(\dots x_{-1}, \underset{\substack{\uparrow \\ 0^{th}\text{ place}}}{x_0}, x_1, x_2, \dots) = (\dots x_{-1}, x_0, \underset{\substack{\uparrow \\ 0^{th}\text{ place}}}{x_1}, x_2, \dots)$$

Let μ = equlibrum state with Hölder potential

Let $(z_1, \dots, z_k) \in K \times \dots \times K$.

Then $\forall f \in C(X, \mathbb{C})$, $\forall x \in X_\mu$,

$$\frac{1}{n} \sum_{j=0}^{n-1} z_{x_0} z_{x_1} \cdots z_{x_{j-1}} \left(f(T^j x) - \int f \, d\mu \right) \to 0$$

This convergence is uniform in $(z_1, \dots, z_k)$.

Let $U_1, \dots, U_k$ be pairwise commuting unitary operators on a Hilbert space H.

Let $v \in H$, $f: X \to \mathbb{C}$ continuous.

Then $\forall x \in X_\mu$

$$\frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) U_{x_0} U_{x_1} \cdots U_{x_{j-1}} v \to \int f \, d\mu \; \Pi_{\langle u \rangle} v$$

where $\Pi_{\langle u \rangle}$ = orthogonal projection onto subspace of vectors fixed by $U_1, \dots, U_k$.

Wiener-Wintner thm using non-abelian skew product

Thm (Santos & W.)

Suppose:
- T cts tx of cpct metric X
- G compact Lie group (wlog G is a closed subgroup of $O(d)$).
- $\Phi : X \longrightarrow G$ cts
- $T_\Phi : X \times G \longrightarrow X \times G$ ergodic wrt $\mu \times$ Haar.

Then: • $\exists\, X_\mu \subset X$, $\mu(X_\mu) = 1$ s.t.

$$(*) \quad \frac{1}{n} \sum_{j=0}^{n-1} \Phi(x)\Phi(Tx)\cdots\Phi(T^{j-1}x)\, f(T^j x) \longrightarrow \pi_{Fix(G)} \int f\, d\mu$$

$$\forall f \in C(X, \mathbb{R}^d),\ \forall x \in X_\mu.$$

where $\pi_{Fix\,G} : \mathbb{R}^d \to Fix(G) = \{ v \in \mathbb{R}^d \mid gv = v\ \forall g \in G \}$ is orthogonal projection.

Remarks

- X_μ is independent of Φ
- T_Φ not ergodic $\Rightarrow (*)$ converges to a limit $l(x) : X \to \mathbb{R}^d$ st $l(Tx) = \Phi(x)\, l(x)$
- The convergence in $(*)$ is uniform in Φ chosen from compact subsets of $C(X, G)$ (when $\int f d\mu = 0$)

Application 1: Random ergodic thm for non-commuting operators.

Let $X = \{1, \ldots, k\}^{\mathbb{Z}}$, $T = $ shift,

$\mu = $ equilibrium state with Hölder potential.

Let $(A_1, \ldots, A_k) \in G \times \cdots \times G$.

Then $\forall f \in C(X, \mathbb{R}^d)$, $\forall x \in X_\mu$

$$\frac{1}{n} \sum_{j=0}^{n-1} A_{x_0} \cdots A_{x_{j-1}} \left(f(T^j x) - \int f \, d\mu \right) \longrightarrow 0$$

uniformly in $(A_1, \ldots, A_k)$.

Let $v \in \mathbb{C}^d$, $f : X \longrightarrow \mathbb{C}$ continuous.

Then $\forall x \in X_\mu$

$$\frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) A_{x_0} \cdots A_{x_{j-1}} v \longrightarrow \int f \, d\mu \; \pi_{\langle A \rangle} v$$

where $\pi_{\langle A \rangle} = $ orthogonal projection onto subspace of vectors fixed by $A_1, \ldots, A_k$.

Application 2: Euclidean extensions

Let
- $T: X \to X$ cts tx of compact metric space
- μ = ergodic Borel probability measure
- $G < O(d)$ closed subgroup
- $\Gamma = G \ltimes \mathbb{R}^d < \mathrm{Isom}(\mathbb{R}^d)$, a subgroup of the Euclidean group

$$(A, u)(B, v) = (AB, u + Av)$$

- $\Phi: X \to G$, $f: X \to \mathbb{R}^d$, $\int f \, d\mu = 0$

Form the (non-compact) skew-product (a Euclidean extension)

$$T_{\Phi, f}: X \times \Gamma \to X \times \Gamma$$
$$(x, y, t) \mapsto (Tx, y\Phi(x), t + yf(x))$$

Then

$$T^n_{\Phi, f}(x, y, t) = \left(T^n x, y\Phi(x) \cdots \Phi(T^{n-1}x), t + y\sum_{j=0}^{n-1} \Phi(x) \cdots \Phi(T^{j-1}x) f(T^j x)\right)$$

W.W $\Rightarrow$ the $\mathbb{R}^d$ component grows sub-linearly a.e.

& this sublinear growth is "stable"

in Φ, f.