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Methods of Integrable Systems in Geometry
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Do we already know enough integrable systems?

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Integrables systems: 1+1 PDEs

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$$u = (u^1, \ldots, u^n)$$

$$n < \infty$$

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- Motivations: integrable PDEs and GW invariants
- What is integrability?
- Critical behaviour in hyperbolic equations and their perturbations
- The classification problem
- Universality in Hamiltonian PDEs
- Further examples and open problems

Original motivations: from 2D TFT / GW invariants

Integrable PDEs $\Longrightarrow$

topological invariants of sophisticated moduli spaces

Example 1 (Witten - Kontsevich) Topological invariants of the moduli spaces $\bar{\mathcal{M}}_{g,n}$ of stable algebraic curves of genus g can be computed from the KdV hierarchy

$$\mathcal{F}(\mathbf{t};\epsilon) = \sum_{g\geq 0} \epsilon^{2g-2}\mathcal{F}_g(\mathbf{t}), \quad \mathcal{F}_g(\mathbf{t}) = \sum \frac{1}{n!} t_{p_1}\dots t_{p_n} \int_{\bar{\mathcal{M}}_{g,n}} \psi_1^{p_1}\wedge\dots\wedge\psi_n^{p_n}$$

then $\quad u = \epsilon^2 \partial_x^2 \mathcal{F}, \quad x = t_0, \quad$ satisfies

$$u_{t_0} = u_x$$

$$u_{t_1} = u\, u_x + \frac{\epsilon^2}{12} u_{xxx}$$

$$u_{t_2} = \frac{1}{2}u^2 u_x + \frac{\epsilon^2}{12}(2u_x u_{xx} + u\, u_{xxx}) + \frac{\epsilon^4}{240} u^V, \quad \dots$$

$$\epsilon \frac{\partial L}{\partial t_k} = [A_k, L]\,, \quad L = \frac{\epsilon^2}{2}\frac{d^2}{dx^2} + u, \quad A_k = \frac{2^{\frac{2k+1}{2}}}{(2k+1)!!}\left(L^{\frac{2k+1}{2}}\right)_+$$

$$\mathcal{F} = \frac{1}{\epsilon^2}\left(\frac{t_0^3}{6} + \frac{t_0^3\,t_1}{6} + \frac{t_0^3\,t_1^2}{6} + \frac{t_0^3\,t_1^3}{6} + \frac{t_0^3\,t_1^4}{6} + \frac{t_0^4\,t_2}{24} + \frac{t_0^4\,t_1\,t_2}{8}\right.$$

$$\left. + \frac{t_0^4\,t_1^2\,t_2}{4} + \frac{t_0^5\,t_2^2}{40} + \frac{t_0^5\,t_3}{120} + \frac{t_0^5\,t_1\,t_3}{30} + \frac{t_0^6\,t_4}{720} + \dots\right)$$

$$+\left(\frac{t_1}{24} + \frac{t_1^2}{48} + \frac{t_1^3}{72} + \frac{t_1^4}{96} + \frac{t_0\,t_2}{24} + \frac{t_0\,t_1\,t_2}{12} + \frac{t_0\,t_1^2\,t_2}{8} + \frac{t_0^2\,t_2^2}{24}\right.$$

$$\left. + \frac{t_0^2\,t_3}{48} + \frac{t_0^2\,t_1\,t_3}{16} + \frac{t_0^3\,t_4}{144} + \dots\right)$$

$$+\epsilon^2\left(\frac{7\,t_2^3}{1440} + \frac{7\,t_1\,t_2^3}{288} + \frac{29\,t_2\,t_3}{5760} + \frac{29\,t_1\,t_2\,t_3}{1440} + \frac{29\,t_1^2\,t_2\,t_3}{576} + \frac{5\,t_0 t_2^2 t_3}{144}\right.$$

$$+ \frac{29\,t_0\,t_3^2}{5760} + \frac{29\,t_0\,t_1\,t_3^2}{1152} + \frac{t_4}{1152} + \frac{t_1\,t_4}{384} + \frac{t_1^2\,t_4}{192} + \frac{t_1^3\,t_4}{96} + \frac{11\,t_0\,t_2\,t_4}{1440}$$

$$\left. + \frac{11\,t_0\,t_1\,t_2\,t_4}{288} + \frac{17\,t_0^2\,t_3\,t_4}{1920} + \dots\right) + O(\epsilon^4).$$

Example 2: (Extended) Toda hierarchy $\Rightarrow$ topological invariants of moduli spaces of stable maps

$$\mathcal{M}_{g,n}(\mathbf{P}^1, \beta) = \left\{ f : (C_g, x_1, \dots, x_n) \to \mathbf{P}^1,\ \beta = \text{degree of the map } f \right\}$$

Difference Lax operator

$$L = \Lambda + v + e^u \Lambda^{-1}, \quad \Lambda = e^{\epsilon \partial_x}$$

$$\epsilon \frac{\partial L}{\partial t_k} = \frac{1}{(k+1)!} \left[(L^{k+1})_+, L \right], \qquad \epsilon \frac{\partial L}{\partial s_k} = \frac{2}{k!} \left[\left(L^k (\log L - c_k) \right)_+, L \right]$$

$$c_k = 1 + \frac{1}{2} + \dots + \frac{1}{k}$$

Associated with the standard **Toda lattice equations**

$$\ddot{u}_n = e^{u_{n-1} - u_n} - e^{u_n - u_{n+1}}, \quad \Leftrightarrow \quad \left. \begin{aligned} \epsilon \partial_{t_0} u &= v(s_0) - v(s_0 + \epsilon) \\ \epsilon \partial_{t_0} v &= e^{u(s_0 + \epsilon)} - e^{u(s_0)} \end{aligned} \right\}$$

$$\log \tau_{\mathbf{P}^1}(s_0, t_0, s_1, t_1, \ldots; \epsilon^2) = \sum_{g \geq 0} \epsilon^{2g-2} \mathcal{F}_g$$

$$\mathcal{F}_g = \sum \frac{1}{n!} t_{\alpha_1, p_1} \ldots t_{\alpha_n, p_n}$$

$$\times \int_{[\bar{\mathcal{M}}_{g,n}(\mathbf{P}^1, \beta)]} \mathrm{ev}_1^* \phi_{\alpha_1} \wedge \psi_1^{p_1} \wedge \ldots \wedge \mathrm{ev}_n^* \phi_{\alpha_n} \wedge \psi_n^{p_n}$$

$$t_{1,p} = s_p, \quad t_{2,p} = t_p$$

Tau-function defined by

$$u = \log \frac{\tau(s_0 + \epsilon)\tau(s_0 - \epsilon)}{\tau^2(s_0)}$$

$$v = \epsilon \frac{\partial}{\partial t_0} \log \frac{\tau(s_0 + \epsilon)}{\tau(s_0)}.$$

Example 3: Toda hierarchy and enumeration of ribbon graphs/triangulations of Riemann surfaces.

Take

$$\log \tau_{\mathbf{P}^1}(s_0, t_0, s_1 + 1, t_1 - 1, s_2, t_2, \dots; \epsilon)|_{t_0=t_1=0,\quad t_k=(k+1)!\lambda_{k+1};\quad s_0=x,\ s_k=0}$$

$$= \frac{x^2}{2\epsilon^2}\left(\log x - \frac{3}{2}\right) - \frac{1}{12}\log x + \sum_{g\geq 2}\left(\frac{\epsilon}{x}\right)^{2g-2}\frac{B_{2g}}{2g(2g-2)}$$

$$+ \sum_{g\geq 0}\epsilon^{2g-2}F_g(x; \lambda_3, \lambda_4, \dots)$$

$$F_g(x; \lambda_3, \lambda_4, \ldots)$$

$$= \sum_n \sum_{k_1,\ldots,k_n} a_g(k_1, \ldots, k_n) \lambda_{k_1} \ldots \lambda_{k_n} x^h,$$

$$h = 2 - 2g - \left(n - \frac{|k|}{2}\right), \quad |k| = k_1 + \ldots + k_n,$$

and

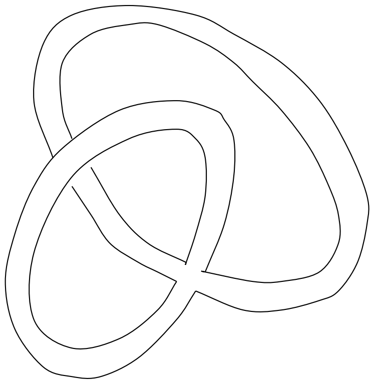
$$a_g(k_1, \ldots, k_n) = \sum_\Gamma \frac{1}{\# \operatorname{Sym} \Gamma}$$

where

Γ = a connected fat graph of genus g

with n vertices of the valencies $k_1, \ldots, k_n$.

E.g.: genus 1, one vertex, valency 4



$$F = \epsilon^{-2}\left[\frac{1}{2}x^2\left(\log x - \frac{3}{2}\right) + 6x^3{\lambda_3}^2 + 2x^3\lambda_4 + 216x^4{\lambda_3}^2\lambda_4 + 18x^4{\lambda_4}^2\right.$$

$$+288x^5{\lambda_4}^3 + 45x^4\lambda_3\lambda_5 + 2160x^5\lambda_3\lambda_4\lambda_5 + 90x^5{\lambda_5}^2 + 5400x^6\lambda_4{\lambda_5}^2 + 5x^4\lambda_6$$

$$+1080x^5{\lambda_3}^2\lambda_6 + 144x^5\lambda_4\lambda_6 + 4320x^6{\lambda_4}^2\lambda_6 + 10800x^6\lambda_3\lambda_5\lambda_6 + 27000x^7{\lambda_5}^2\lambda_6$$

$$\left.+300x^6{\lambda_6}^2 + 21600x^7\lambda_4{\lambda_6}^2 + 36000x^8{\lambda_6}^3\right]$$

$$-\frac{1}{12}\log x + \frac{3}{2}x{\lambda_3}^2 + x\lambda_4 + 234x^2{\lambda_3}^2\lambda_4 + 30x^2{\lambda_4}^2 + 1056x^3{\lambda_4}^3 + 60x^2\lambda_3\lambda_5$$

$$+6480x^3\lambda_3\lambda_4\lambda_5 + 300x^3{\lambda_5}^2 + 32400x^4\lambda_4{\lambda_5}^2 + 10x^2\lambda_6 + 3330x^3{\lambda_3}^2\lambda_6$$

$$+600x^3\lambda_4\lambda_6 + 31680x^4{\lambda_4}^2\lambda_6 + 66600x^4\lambda_3\lambda_5\lambda_6 + 283500x^5{\lambda_5}^2\lambda_6$$

$$+2400x^4{\lambda_6}^2 + 270000x^5\lambda_4{\lambda_6}^2 + 696000x^6{\lambda_6}^3$$

$$+\epsilon^2\left[-\frac{1}{240x^2} + 240x{\lambda_4}^3 + 1440x\lambda_3\lambda_4\lambda_5 + \frac{1}{2}165x{\lambda_5}^2 + 28350x^2\lambda_4{\lambda_5}^2\right.$$

$$+675x{\lambda_3}^2\lambda_6 + 156x\lambda_4\lambda_6 + 28080x^2{\lambda_4}^2\lambda_6 + 56160x^2\lambda_3\lambda_5\lambda_6 + 580950x^3{\lambda_5}^2\lambda_6$$

$$\left.+2385x^2{\lambda_6}^2 + 580680x^3\lambda_4{\lambda_6}^2 + 2881800x^4{\lambda_6}^3\right] + \ldots$$

Other examples:

• Moduli spaces of spin-N structures on Riemann surfaces and Drinfeld - Sokolov hierarchy of A_{N-1} type.

Lax operator

$$L = (\epsilon\partial_x)^N + u_1(x)(\epsilon\partial_x)^{N-1} + \ldots + u_N(x).$$

• Orbifold GW invariants of weighted projective lines and generalized Toda hierarchy.

Lax operator

$$L = \Lambda^p + u_1(x)\Lambda^{p-1} + \ldots + u_p(x) + \ldots + u_{p+q}(x)\Lambda^{-q}, \quad \Lambda = e^{\epsilon\,\partial_x}$$

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Various definitions of integrability:

- Lax operators
- bihamiltonian recursion
- Hirota bilinear relations, infinite-dimensional Grassmanians
- maximal Abelian subalgebras in Hamiltonian vector fields

Main subject: Hamiltonian perturbations of hyperbolic PDEs

$$u_t^i + A_j^i(u)u_x^j + \text{higher order derivatives} = 0, \quad i = 1, \dots, n$$

Weak dispersion expansion: start from

$$u_t^i + F^i(u, u_x, u_{xx}, \dots) = 0$$

Introduce **slow variables** $x \mapsto \epsilon\, x, \quad t \mapsto \epsilon\, t$

$$u_t^i + \frac{1}{\epsilon} F^i(u, \epsilon\, u_x, \epsilon^2 u_{xx}, \dots)$$

$$= u_t^i + A_j^i(u)u_x^j + \epsilon\left(B_j^i(u)u_{xx}^j + \frac{1}{2}C_{jk}^i(u)u_x^j u_x^k\right) + \dots = 0$$

Example. The simplest hyperbolic equation

$$v_t + a(v)v_x = 0$$

is a Hamiltonian PDE,

$$v_t + \{H, v(x)\} = v_t + \partial_x \frac{\delta H}{\delta v(x)} = 0, \quad H = \int f(v)\,dx$$

$$\{v(x), v(y)\} = \delta'(x-y), \qquad\qquad f''(v) = a(v)$$

Any two such flows commute:

$$v_t + a(v)v_x = 0$$

$$(v_t)_s = (v_s)_t$$

$$v_s + b(v)v_x = 0$$

e.g., from commuting Hamiltonians $\{H_f, H_g\} = 0$,

$$H_f = \int f(v)\,dx, \quad H_g = \int g(v)\,dx, \quad f''(v) = a(v), \quad g''(v) = b(v)$$

This is a **complete family** of commuting Hamiltonians!

The solution $v = v(x, t)$ to a Cauchy problem exists till the time $t = t_C$ of **gradient catastrophe**

Point of *gradient catastrophe* $x = x_0$, $t = t_0$, $v = v_0$,

$$v(x,t) \to v_0, \quad v_x(x,t) \to \infty \quad \text{for } (x,t) \to (x_0, t_0), \quad t < t_0$$

Lemma 1 Up to shifts, Galilean transformations and rescalings near the point of gradient catastrophe the generic solution approximately behaves as the root $v = v(x, t)$ of cubic equation

$$x = v\,t - \frac{v^3}{6}$$

(universal unfolding of A_2 singularity)

Proof. The solution can be found by the *method of characteristics*:

$$x = a(v)\,t + b(v) \tag{1}$$

for an arbitrary smooth function $b(v)$. At the point of gradient catastrophe one has

$$\begin{aligned} x_0 &= a(v_0)t_0 + b(v_0) \\ 0 &= a'(v_0)t_0 + b'(v_0) \\ 0 &= a''(v_0)t_0 + b''(v_0) \end{aligned} \tag{2}$$

(inflection point). The **genericity assumption**

$$\kappa := -\left(a'''(v_0)t_0 + b'''(v_0)\right) \neq 0. \tag{3}$$

Introduce the new variables

$$\bar{x} = x - a_0(t - t_0) - x_0$$
$$\bar{t} = t - t_0$$
$$\bar{v} = v - v_0.$$

Here $a_0 = a(v_0)$, $a_0' := a'(v_0)$ etc. Rescaling:

$$\bar{x} \mapsto \lambda\, \bar{x}$$

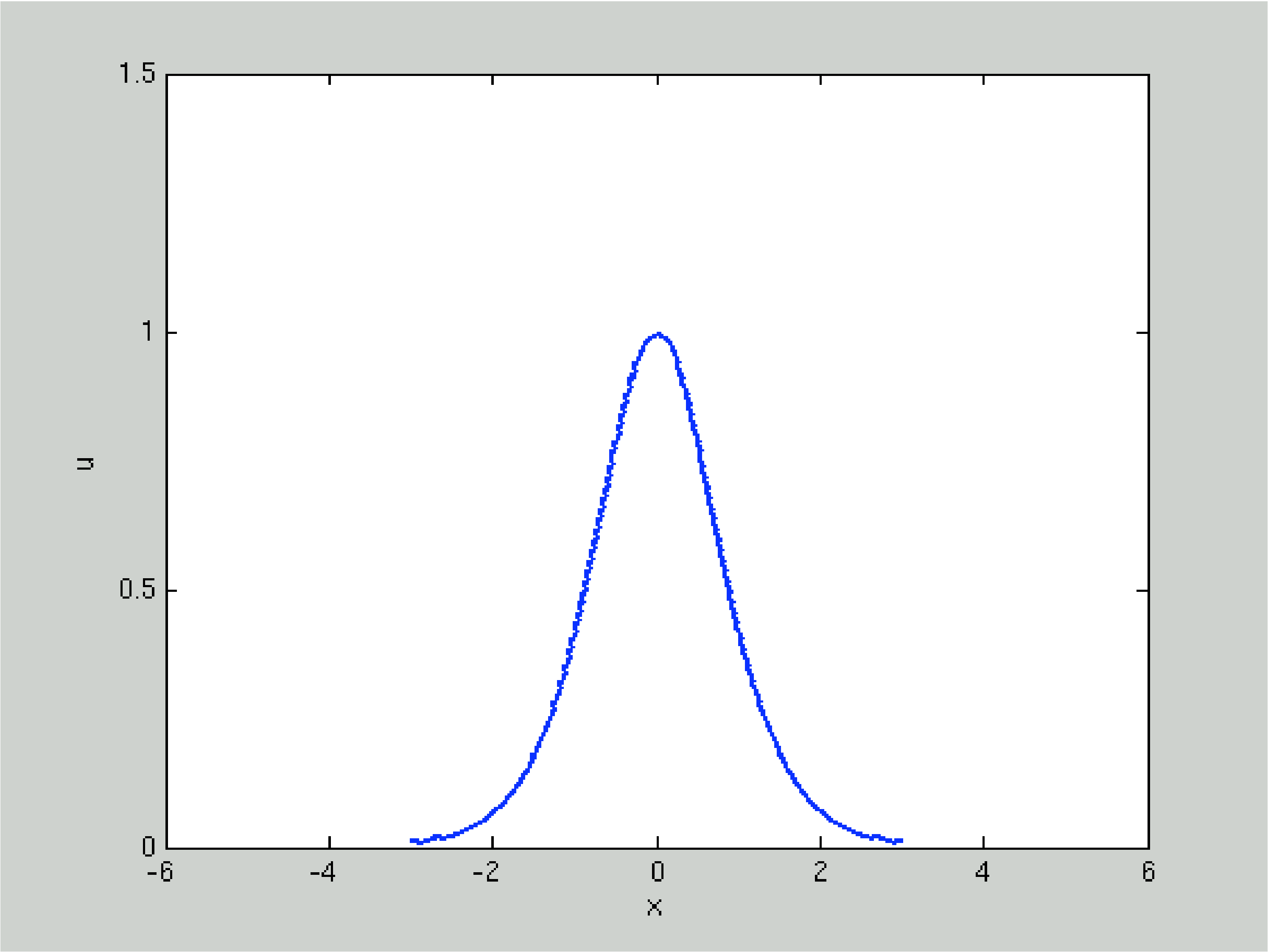
$$\bar{t} \mapsto \lambda^{\frac{2}{3}}\, \bar{t}$$

$$\bar{v} \mapsto \lambda^{\frac{1}{3}}\, \bar{v}$$

Substituting in $x = a(v)\, t + b(v)$ and expanding at $\lambda \to 0$ one obtains, after division by λ

$$\bar{x} = a_0' \bar{t}\, \bar{v} - \frac{1}{6} \kappa\, \bar{v}^3 + O\left(\lambda^{\frac{1}{3}}\right)$$

Gradient catastrophe in $u_t + u\,u_x = 0$



Perturbations: two scenarios

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- Dissipative perturbation: shock waves

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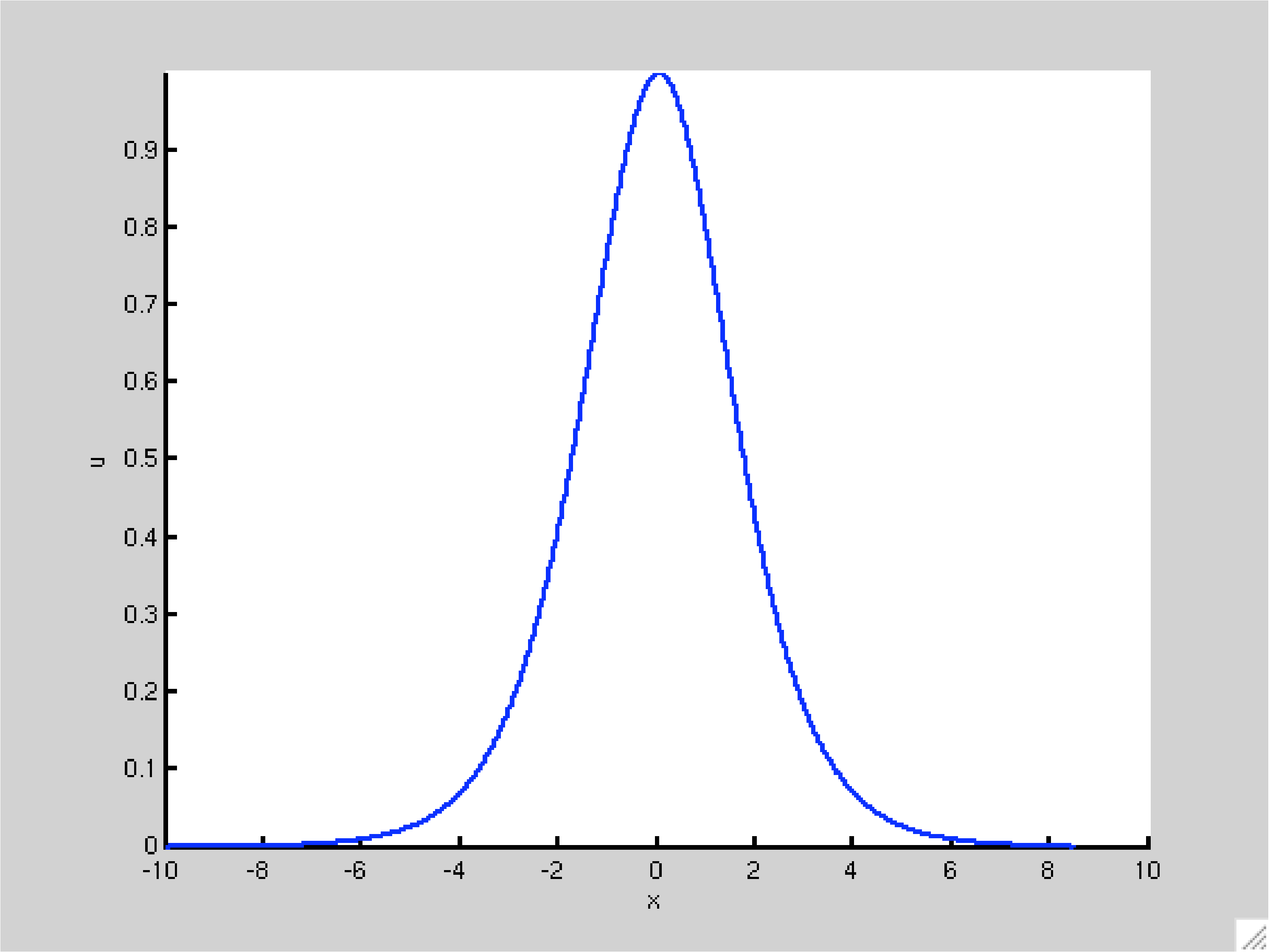
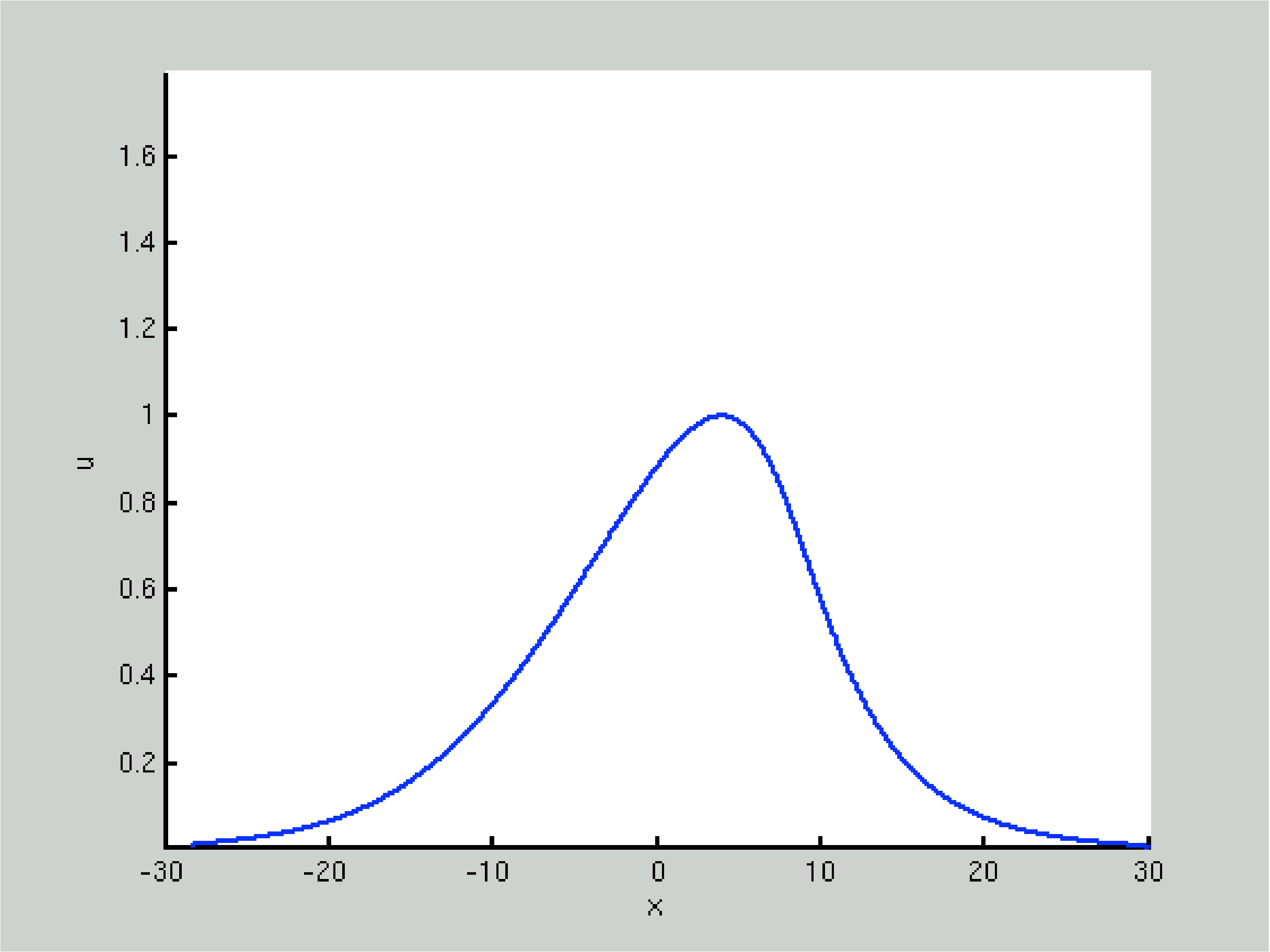
- Hamiltonian perturbations: oscillations

Perturbations: Burgers equation $u_t + u\,u_x = \epsilon\,u_{xx}$ (dissipative case)



Perturbations: KdV equation $u_t + u\,u_x + \epsilon^2 u_{xxx} = 0$

(Hamiltonian case)



The subclass: Hamiltonian perturbations

Main questions:

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- Classification

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- Properties of solutions

Recall:

Hamiltonian hyperbolic systems read (B.D., S.Novikov, 1983)

$$u^i_t + \partial_x \left(\eta^{ij} \frac{\partial h(u)}{\partial u^j} \right) = 0, \quad \eta^{ji} = \eta^{ij}, \quad \det(\eta^{ij}) \neq 0$$

equivalently

$$u^i_t + \{u^i(x), H\} = 0, \quad H = \int h(u)\, dx, \quad \{u^i(x), u^j(y)\} = \eta^{ij} \delta'(x-y).$$

Metric $\quad ds^2 = \eta_{ij} du^i du^j, \quad (\eta_{ij}) = (\eta^{ij})^{-1}$

The goal: classify Hamiltonian perturbations

$$\mathbf{u^i_t + A^i_j(u) u^j_x} + \epsilon \left(B^i_j(u) u^j_{xx} + \frac{1}{2} C^i_{jk}(u) u^j_x u^k_x \right) + \ldots = 0$$

with respect to the group of **Miura-type tranformations**

$$u^i \mapsto \tilde{u}^i = \sum_{k=0}^{\infty} \epsilon^k F^i_k(u; u_x, \ldots, u^{(k)}), \quad i = 1, \ldots, n$$

$$F^i_k \quad \text{a polynomial in} \quad u_x, u_{xx}, \ldots, \quad \deg F^i_k = k,$$

$$\det \left(\frac{\partial F^i_0(u)}{\partial u^j} \right) \neq 0.$$

Definition. The perturbation is called **trivial** if it can be eliminated by a Miura-type transformation

Simple example: Hamiltonian perturbations of Hopf equation

$$v_t + v\, v_x = 0$$

Theorem 2. Any Hamiltonian perturbation of Hopf equation remains **integrable** up to the order $O(\epsilon^4)$.

Step 1: classification. Any perturbation is equivalent, modulo $O(\epsilon^5)$, to one of the form

$$u_t + u\, u_x + \frac{\epsilon^2}{24}\left[2c\, u_{xxx} + 4c' u_x u_{xx} + c'' u_x^3\right] + \epsilon^4 \left[2p\, u_{xxxxx}\right.$$

$$\left. + 2p'(5u_{xx}u_{xxx} + 3u_x u_{xxxx}) + p''(7u_x u_{xx}^2 + 6u_x^2 u_{xxx}) + 2p''' u_x^3 u_{xx}\right] = 0$$

where $c = c(u)$, $p = p(u)$ are two arbitrary functions.

Main arguments:

• *rigidity* of the G-FZ Poisson bracket $\{v(x), v(y)\} = \delta'(x-y)$ (triviality of the Poisson cohomology, Getzler 2001)

So, it suffices to classify *deformations of the Hamiltonian*

$$H_0 = \frac{1}{6}\int v^3\,dx \quad \mapsto \quad H_\epsilon = H_0 + \epsilon H_1 + \epsilon^2 H_2 + \dots$$

• Classify H_ϵ modulo *canonical transformations*

$$u \mapsto u + \epsilon\{u(x), F\} + \frac{\epsilon^2}{2}\{\{u(x), F\}, F\} + \dots$$

(time-ϵ shift generated by a Hamiltonian F).

So, any perturbation of H_0 is equivalent to

$$H_\epsilon = \int \left[\frac{u^3}{6} - \epsilon^2 \frac{c(u)}{24} u_x^2 + \epsilon^4 p(u) u_{xx}^2\right] dx + O(\epsilon^5)$$

for some functions $c(u)$, $p(u)$.

Step 2: deforming the entire commutative algebra. Define a deformations of the first integral $H_0^f = \int f(v)\, dx$ by

$$H_\epsilon^f = \int h_f \, dx$$

$$h_f = f - \frac{\epsilon^2}{24} c\, f''' u_x^2 + \epsilon^4 \left[\left(p\, f''' + \frac{c^2\, f^{(4)}}{480}\right) u_{xx}^2 - \left(\frac{c\, c''\, f^{(4)}}{1152} + \frac{c\, c'\, f^{(5)}}{1152} + \frac{c^2\, f^{(6)}}{3456} + \frac{p'\, f^{(4)}}{6} + \frac{p\, f^{(5)}}{6}\right) u_x^4\right]$$

Then

$$\{H_\epsilon^f, H_\epsilon^g\} = 0 \ \left(\text{mod } O(\epsilon^6)\right) \quad \text{for any} \quad f = f(u),\ g = g(u)$$

Example 1. For $c(u) = \text{const}$, $p(u) = 0$ one obtains the KdV equation

$$u_t + u\,u_x + c\frac{\epsilon^2}{12}u_{xxx} = 0.$$

Example 2. For $c(u) = 8\,u$, $p(u) = \frac{1}{3}u \Rightarrow$ Camassa-Holm equation

$$u_t = (1 - \epsilon^2\partial_x^2)^{-1}\left\{\frac{3}{2}u\,u_x - \epsilon^2\left[u_x u_{xx} + \frac{1}{2}u\,u_{xxx}\right]\right\}$$

Example 3. The case

$$c(u) = 2, \quad p(u) = -\frac{1}{240}$$

corresponds to the Volterra lattice

$$\dot{q}_n = q_n(q_{n+1} - q_{n-1}), \quad q_n = e^{u(n\,\epsilon)}.$$

Implications (?):

Conjecture 3: **all** generic solutions of **any** generic Hamiltonian perturbations of

$$v_t + v\, v_x = 0$$

have the **same**, up to shifts, rescalings and Galilean transformations, universal critical behaviour.

The same behaviour for the solutions to **any** of the perturbed commuting flows

$$v_s + a(v)\, v_x = 0$$

Step 1: behaviour *before* the critical point

Quasitriviality: there exists a canonical transformation

$$v \mapsto u = v + \epsilon^2 F_2(v; v_x, v_{xx}, v_{xxx}) + \epsilon^4 F_4(v; v_x, \dots, v^{(6)}) + \dots$$

rational in derivatives intertwinning between the perturbed and unperturbed families of commuting PDEs. Explicitly:

$$v \mapsto v + \epsilon\{v(x), K\} + \frac{\epsilon^2}{2}\{\{v(x), K\}, K\} + \dots$$

with

$$K = \int \left[-\frac{1}{24}\epsilon\, c(v)\, v_x \log v_x - \epsilon^3 \left(\frac{c^2(v)}{5760} \frac{v_{xx}^3}{v_x^3} - \frac{p(v)}{4} \frac{v_{xx}^2}{v_x} \right) \right] dx,$$

That is, one obtains **any** (formal) solution $u(x, t; \epsilon)$ to

$$u_t + \partial_x \frac{\delta H_\epsilon^f}{\delta u(x)} = O(\epsilon^5)$$

by the substitution

$$\begin{aligned}
v \mapsto u = v + \frac{\epsilon^2}{24}\partial_x \left(c\frac{v_{xx}}{v_x} + c' v_x\right) + \epsilon^4 \partial_x \Bigg[c^2 \left(\frac{v_{xx}^3}{360\, v_x^4} - \frac{7\, v_{xx} v_{xxx}}{1920\, v_x^3} + \frac{v_{xxxx}}{1152\, v_x^2}\right)_x \\
+ c\, c' \left(\frac{47\, {v_{xx}}^3}{5760\, {v_x}^3} - \frac{37\, v_{xx}\, v_{xxx}}{2880\, {v_x}^2} + \frac{5\, v_{xxxx}}{1152\, v_x}\right) + c'^2 \left(\frac{v_{xxx}}{384} - \frac{{v_{xx}}^2}{5760\, v_x}\right) + c\, c'' \left(\frac{v_{xxx}}{144} - \frac{{v_{xx}}^2}{360\, v_x \cdots}\right. \\
+ \frac{1}{1152}\left(7\, c'\, c''\, v_x\, v_{xx} + c''^2\, {v_x}^3 + 6\, c\, c'''\, v_x\, v_{xx} + c'\, c'''\, {v_x}^3 + c\, c^{(4)}\, {v_x}^3\right) \\
+ p \left(\frac{{v_{xx}}^3}{2\, {v_x}^3} - \frac{v_{xx}\, v_{xxx}}{{v_x}^2} + \frac{v_{xxxx}}{2\, v_x}\right) + p' v_{xxx} + p'' \frac{v_x\, v_{xx}}{2} \Bigg] \qquad (1
\end{aligned}$$

applied to a solution $v = v(x, t)$ of

$$v_t + a(v) v_x = 0, \quad a(v) = f''(v).$$

Step 2: Introducing a *special function*.

Consider the following fourth order ODE for the function $U = U(X)$ depending on T as on the parameter (**ODE4**):

$$X = T\,U - \left[\frac{1}{6}U^3 + \frac{1}{24}\left(U'^2 + 2U\,U''\right) + \frac{1}{240}U^{IV}\right].$$

Main Conjecture. 1). The 4th order ODE has **unique** solution $U = U(X;T)$ **smooth for all real** $X \in \mathbb{R}$ for all values of the parameter T.

Let us call the solution to the perturbed PDE **generic** if, along with the condition $\kappa = -\left(a'''(v_0)t_0 + b'''(v_0)\right) \neq 0$ it also satisfies

$$c_0 := c(v_0) \neq 0.$$

3). The above solution can be extended up to $t = t_0$; near the point (x_0, t_0) it behaves in the following way

$$u \simeq v_0 + \left(\frac{\epsilon^2 c_0}{\kappa^2}\right)^{1/7} U\left(\frac{x - a_0(t-t_0) - x_0}{(\kappa\, c_0^3 \epsilon^6)^{1/7}};\ \frac{a_0'(t-t_0)}{(\kappa^3 c_0^2 \epsilon^4)^{1/7}}\right) + O\left(\epsilon^{4/7}\right).$$

“Proof” of the formula is obtained by rescaling

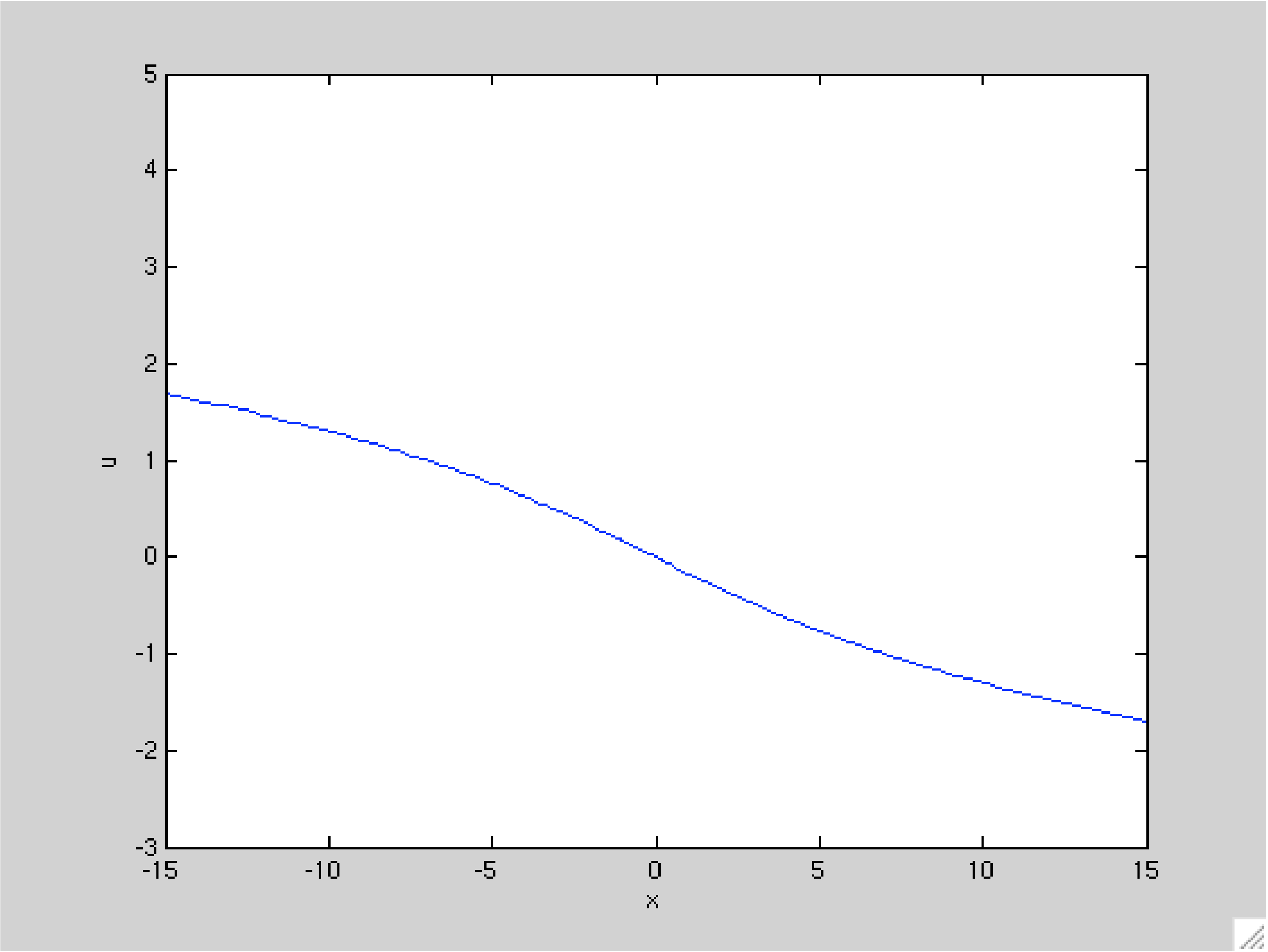
$$\bar{x} \mapsto \lambda\, \bar{x}$$

$$\bar{t} \mapsto \lambda^{\frac{2}{3}}\, \bar{t}$$

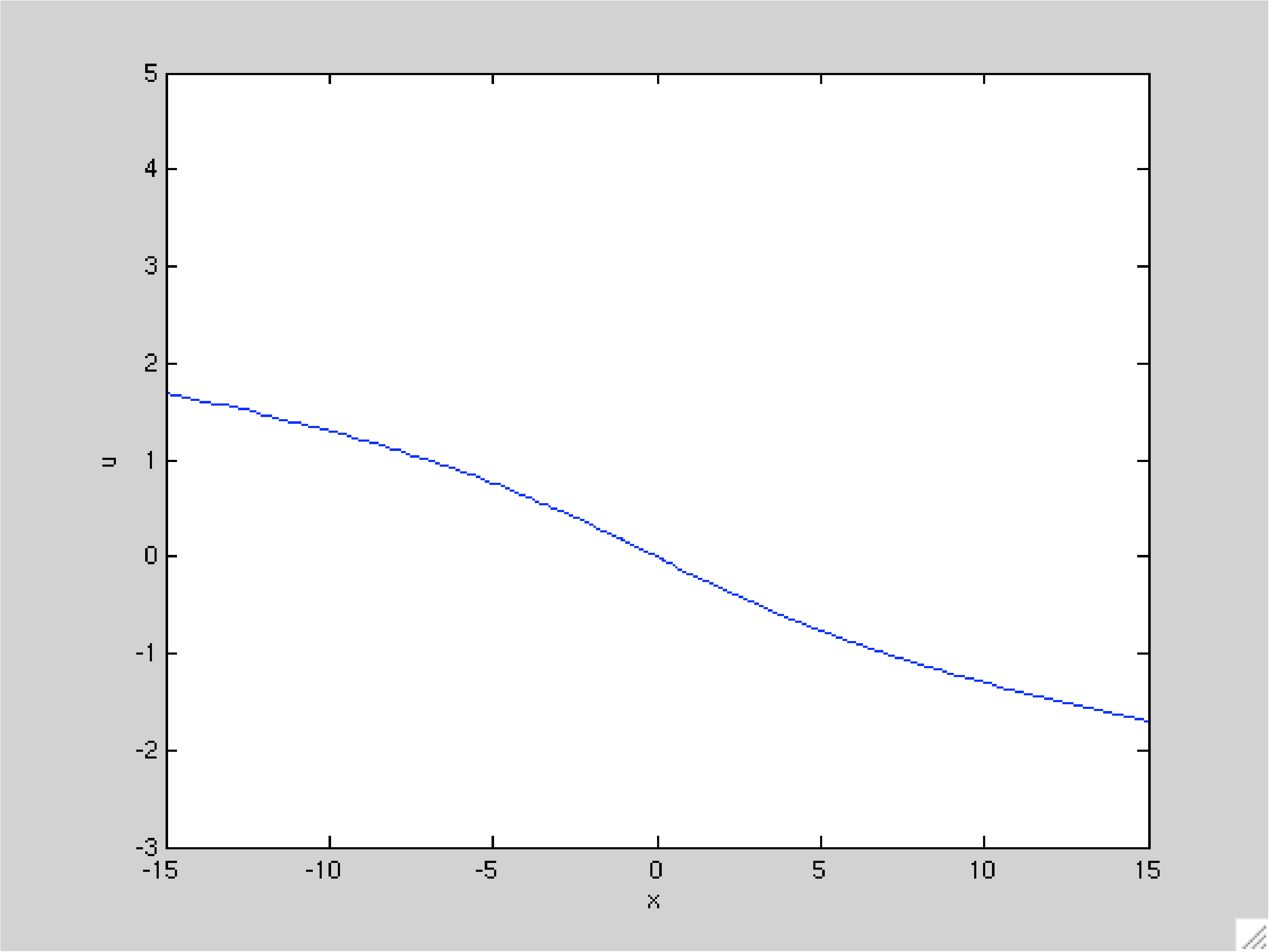
$$\bar{v} \mapsto \lambda^{\frac{1}{3}}\, \bar{v}$$

$$\epsilon \mapsto \lambda^{7/6}\epsilon.$$

A_2 singularity



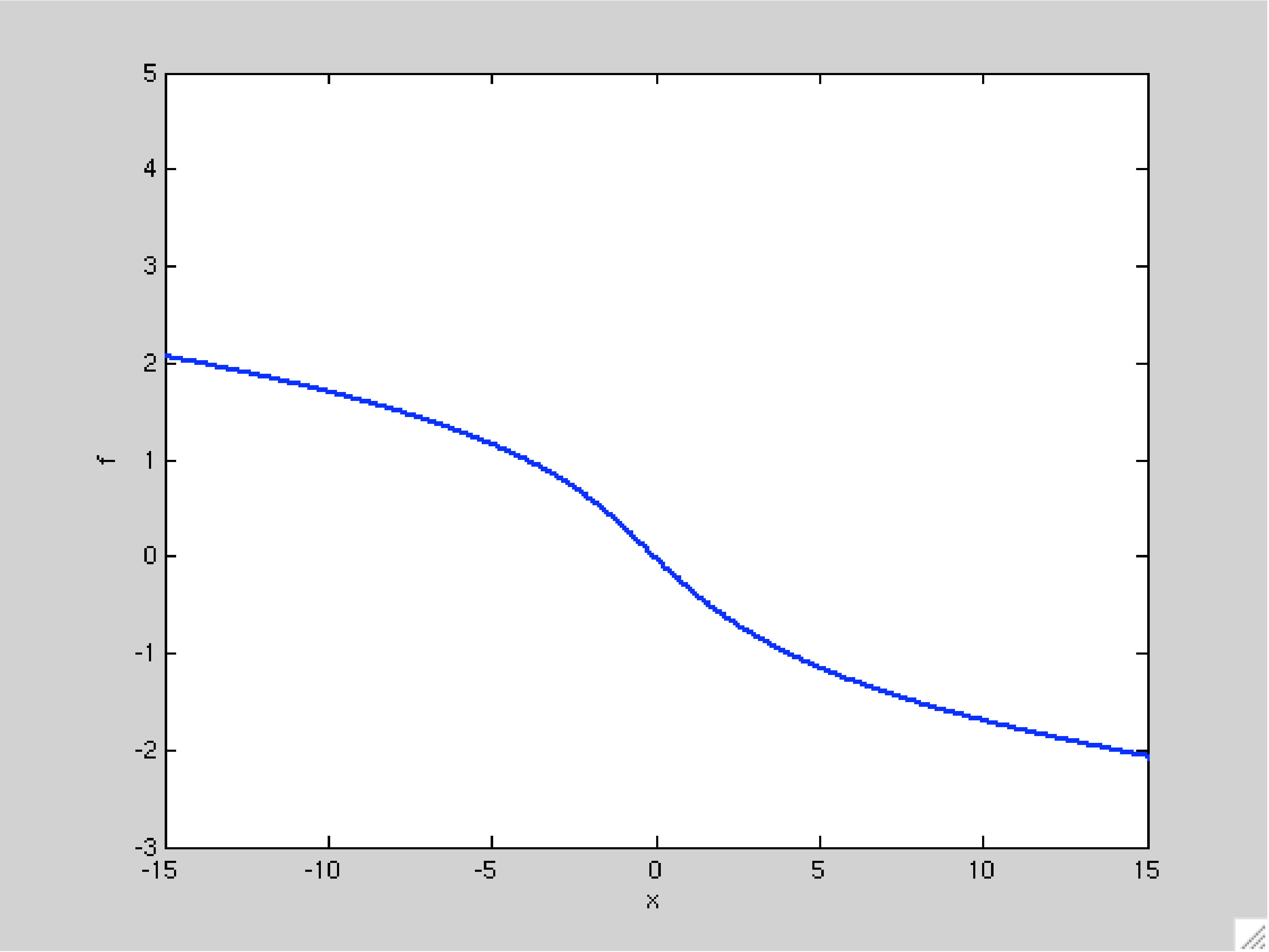
A_2 singularity



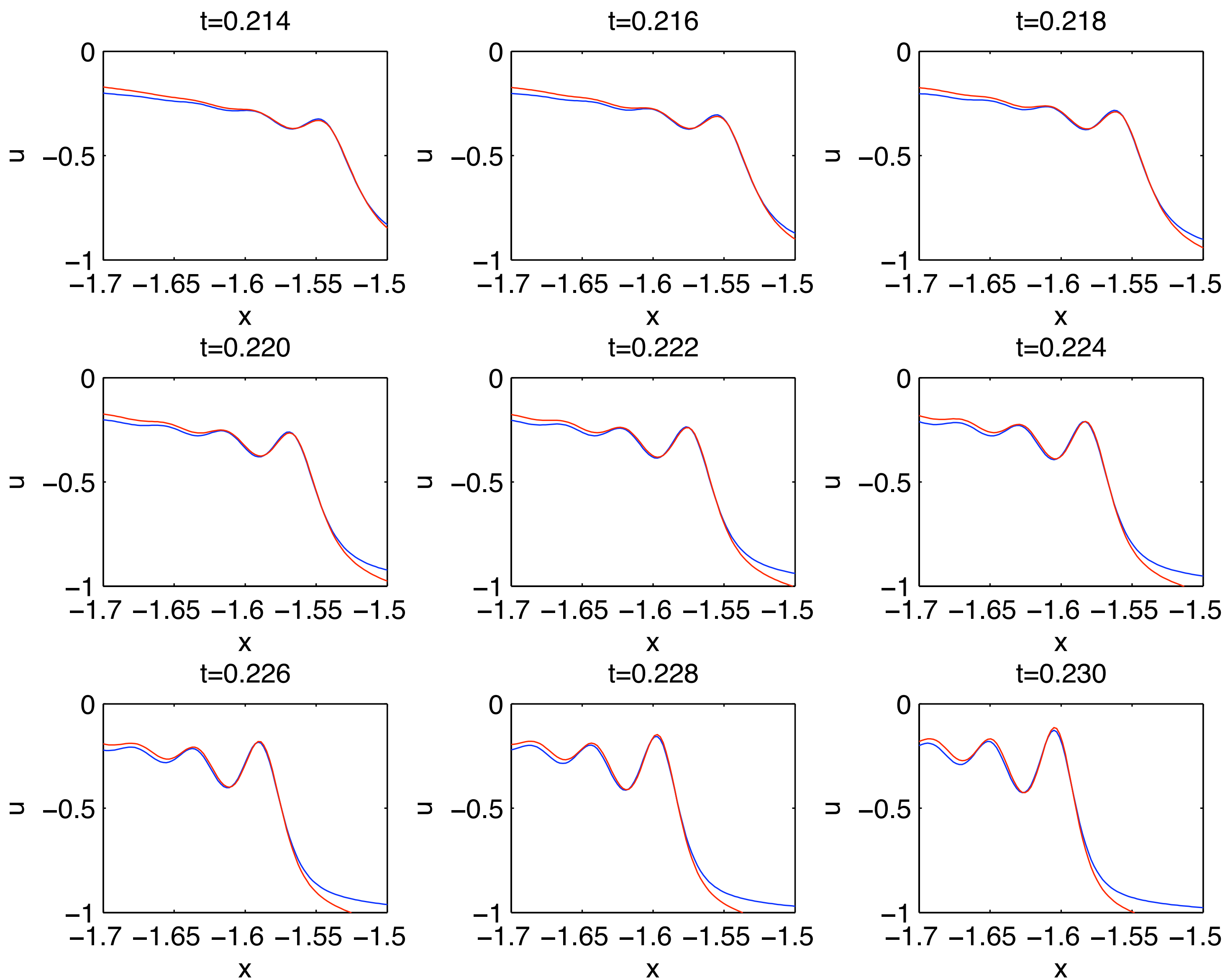
to be replaced by

Smooth solution to the ODE4

$$X = T\,U - \left[\frac{1}{6}U^3 + \frac{1}{24}\left(U'^2 + 2U\,U''\right) + \frac{1}{240}U^{IV}\right]$$



KdV versus ODE4



The conjectural existence of the smooth solution to the ODE4 has been first discussed by Brézin, Marinari, Parisi and by Moore in 1990 (for the particular value $T = 0$) in the setting of the theory of random matrices.

Importance of the smooth solution to the ODE4 for the so-called Gurevich - Pitaevsky solution to KdV was discussed by Suleimanov (1994) and Kudashev and Suleimanov (1996).

Existence of a smooth solution to ODE4 was recently proved by T.Claeys and M.Vanlessen, April 2006, using the technique of Riemann - Hilbert problem. Also the asymptotics $U \sim (-6\,X)^{1/3}$ for $|X| \to \infty$ has been established. Within this class the uniqueness can be established using results of Moore (1990) and Menikoff (1972)

Generalization for systems? $n = 2$, $u = u(x,t)$, $v = v(x,t)$. Existence of a catastrophe: Klainerman, Majda (1980).

Local behaviour: Whitney singularity

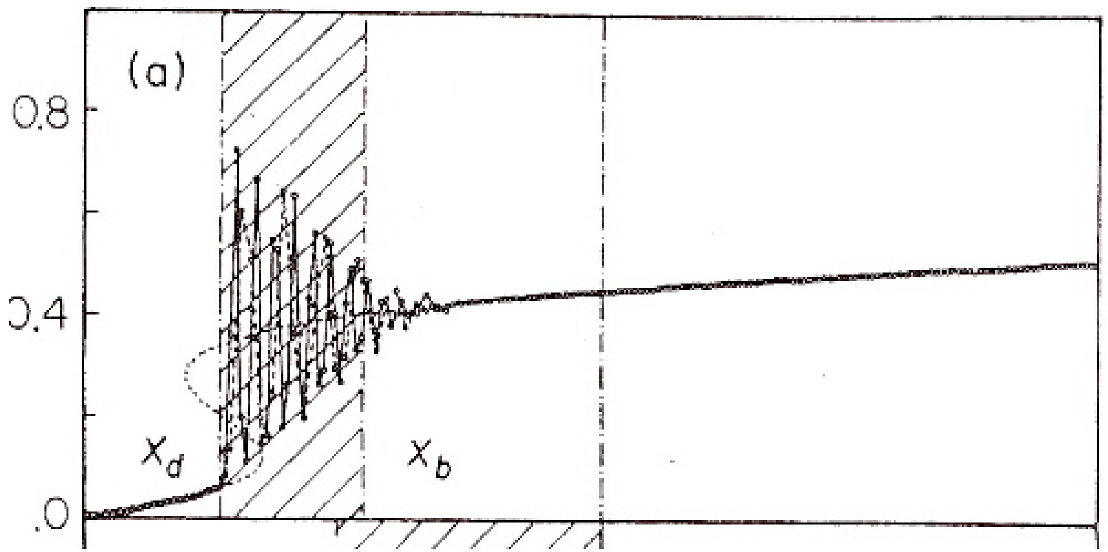
$$x_+ = r_+$$

$$x_- = r_+ r_- - \frac{1}{6} r_-^3$$

(by a nonlinear/linear change of dependent/independent variables $r_\pm = r_\pm(u,v)$, $x_\pm = a_\pm(x - x_0) + b_\pm(t - t_0)$).

After the perturbation?

Cf: (1) oscillatory behaviour of correlation functions in the random matrix models:



(from Jurkiewicz, Phys. Lett. B, 1991). Hamiltonian perturbations of dispersionless Toda hierarchy

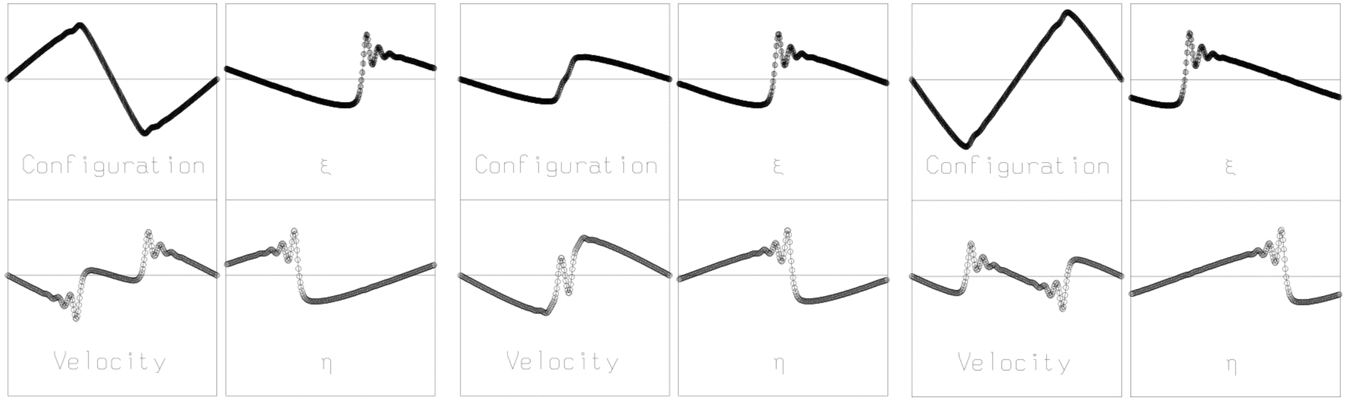
Rigorous results (Claeys, Vanlessen, July 2006): asymptotics in Hermitean random matrices near singular edge points: for the recurrence coefficients

$$a_n(s,t) = a_n^0 + \tfrac{1}{2} c\, n^{-2/7} U(c_1 n^{6/7} s, c_2 n^{4/7} t) + O\left(n^{-3/7}\right)$$

$$b_n(s,t) = b_n^0 + c\, n^{-2/7} U(c_1 n^{6/7} s, c_2 n^{4/7} t) + O\left(n^{-3/7}\right)$$

Fermi - Pasta - Ulam numerical experiments:

$$H = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + V\left(q_{i+1} - q_i\right)$$



(from Lorenzoni, Paleari, nlin/0511026). Hamiltonian perturbations of

$$\begin{aligned} u_t &= v_x \\ v_t &= V''(u) u_x \end{aligned}$$

Further programme:

- table of singularities of solutions to hyperbolic equations (algebraic functions)

- table of singularities of solutions to perturbed systems (Painlevé transendents, theta functions, . . .)

- matching problem

Multicomponent case: deformation theory of *bihamiltonian* hyperbolic PDEs.

• Uniqueness results ("bihamiltonian cohomology"): any deformation of an order n hyperbolic system depends on at most n arbitrary functions of 1 variable $c_1(u_1), \ldots, c_n(u_n)$

• Existence result (in progress): for the integrable hyperbolic systems associated with a semisimple Frobenius manifold the integrable deformation exists in all orders in ϵ for the particular choice

$$c_1 = c_2 = \ldots = c_n = 1$$

(integrable hierarchies of the topological type).

The construction uses “quantization” of the Riemann - Hilbert problem associated with the semisimple Frobenius manifold considered as a canonical transformation of the Givental symplectic space.

Corollary. For any n the total GW potential of $\mathbf{CP}^n$ is a tau function of a particular solution to an integrable hierarchy of the order $n+1$. (Uses Givental’s proof of the Virasoro conjecture for $\mathbf{CP}^n$).

Frobenius Manifold	Orbit spaces, Finite Coxeter groups	Orbit spaces, Extended affine Weyl groups	Orbit spaces, Jacobi groups	Hurwitz spaces	$QH^*(\mathbf{P}^n)$, $QH^*(G_{m,n})$, $\ldots$	Singularit unfolding
Hierarchy	ADE Drinfeld -Sokolov	$\tilde{A}_1$ Toda $\tilde{A}_{k,l}$ bigraded Toda	?	$g=0$ reductions of nKP	?	?
Applications	A_1 W-K A_n spin$(n+1)$ structures	$\tilde{A}_1$ $GW(\mathbf{P}^1)$ $\tilde{A}_{k,l}$ – $GW(\mathbf{P}^1_{k,l})$	?	Higher order Whitham theory	$GW(\mathbf{P}^n), \ldots$ $g \geq 0$	?

References:

S.-Q. Liu, Y. Zhang, Deformations of semisimple bihamiltonian structures of hydrodynamic type, J. Geom. Phys. **54** (2005) 427–453.

B.D., S-Q.Liu, Y.Zhang, On Hamiltonian perturbations of hyperbolic systems of conservation laws I: quasi-triviality of bi-hamiltonian perturbations, *Comm. Pure Appl. Math.* **59** (2006) 559-615

B.D., On Hamiltonian perturbations of hyperbolic systems of conservation laws II: universality of critical behaviour, *Comm. Math. Phys.*, on-line April 2006.

Numerics:

courtesy of T. Grava and C.Klein

Thank you!