Some examples of 'second order elliptic integrable systems associated to a 4-symmetric space'

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1 Hamiltonian Stationary Lagrangian (HSL) surfaces

1.1 A variational problem in $\mathbb{R}^4$

$\mathbb{R}^4$ has the canonical Euclidean structure $\langle\cdot,\cdot\rangle$ and the symplectic form $\omega := dx^1\wedge dx^2 + dx^3\wedge dx^4$. An immersed surface $\Sigma\subset\mathbb{R}^4$ is

(i) **Lagrangian** iff $\omega_{|\Sigma}=0$

(ii) **Hamiltonian Stationary Lagrangian** (HSL) iff $\omega_{|\Sigma}=0$ and Σ is a critical point of the area functional $\mathcal{A}$ with respect to all *Hamiltonian vector fields* ξ_h s.t.:

- $\exists h\in\mathcal{C}^\infty_c(\mathbb{R}^4)$, $\xi_h \lrcorner\, \omega + dh = 0$
- equivalentely, if J is the complex structure s.t. $\omega = \langle J\cdot,\cdot\rangle$, $\xi_h = J\nabla h$.

It means that $\delta\mathcal{A}_\Sigma(\xi_h)=0$, $\forall h\in\mathcal{C}^\infty_c(\mathbb{R}^4)$.

What is the Euler equation ?
The Gauss map is:

$$\begin{array}{cccccc}\gamma:\Sigma & \longrightarrow & Gr_{Lag}(\mathbb{R}^4) & \subset & Gr_2(\mathbb{R}^4)\\ & & S^1\times S^2 & \subset & S^2\times S^2\end{array}$$

Denote $\gamma=(\rho_\Sigma,\sigma_\Sigma)$ the two components of γ. For a Lagrangian immersion $\rho_\Sigma\simeq e^{i\beta}$. Then the *mean curvature vector* is

$$\vec{H} = J\nabla\beta.$$

Lemma 1.1 *Σ is HSL iff*

$$\left\{\begin{array}{c}\omega_{|\Sigma}=0\\ \Delta_\Sigma\beta=0.\end{array}\right.$$

Remark: Σ is special Lagrangian iff $\left\{ \begin{array}{c} \omega_{|\Sigma} = 0 \\ \beta = \text{Constant.} \end{array} \right. \iff \left\{ \begin{array}{c} \omega_{|\Sigma} = 0 \\ \Sigma \text{ is minimal} \end{array} \right.$.

An analytic study was done by R. Schoen and J. Wolfson [6] (in a 4-dimensional Calabi–Yau manifold).

1.2 It is a completely integrable system (F.H.–P. Romon [1, 2])

Let $\Omega \subset \mathbb{C}$ be an open subset and $X : \Omega \longrightarrow \mathbb{R}^4$ a (local) conformal parametrization of Σ. Set

$$\rho_X := \rho_\Sigma \circ X,$$

the *left Gauss map*.

Idea: to lift the pair (X, ρ_X) to a map $F : \Omega \longrightarrow \mathfrak{G}$, where $\mathfrak{G}$ is a local symmetry group of the problem. The more naive choice is $\mathfrak{G} = SO(4) \ltimes \mathbb{R}^4$, the group of isometries of $\mathbb{R}^4$. Then

$$F = \begin{pmatrix} R & X \\ 0 & 1 \end{pmatrix} \simeq (R, X),$$

where $R : \Omega \longrightarrow SO(4)$ encodes $\rho_X \simeq e^{i\beta}$. (Alternatively one can choose $\mathfrak{G} = U(2) \ltimes \mathbb{C}^2$, with the identification $\mathbb{C}^2 \simeq (\mathbb{R}^4, J)$ and $U(2)$: subgroup of $SO(4)$. Then the way $R \in U(2)$ encodes β is simply through the relation $\det_{\mathbb{C}} R = e^{i\beta}$).

In all cases there exists an automorphism $\tau : \mathfrak{G} \longrightarrow \mathfrak{G}$ s.t. $\tau^4 = Id$. This automorphism acts on the Lie algebra $\mathfrak{g}$ and can be diagonalized with the eigenvalues i, 1, $-i$ and -1. Hence the vector space decomposition

$$\begin{array}{ccccccccc} \mathfrak{g}^{\mathbb{C}} = & \mathfrak{g}_{-1} & \oplus & \mathfrak{g}_0^{\mathbb{C}} & \oplus & \mathfrak{g}_1 & \oplus & \mathfrak{g}_2^{\mathbb{C}} \\ \text{eigenvalues:} & -i & & 1 & & i & & -1 \end{array}$$

Then consider the (pull-back of the) Maurer–Cartan form

$$\alpha := F^{-1} dF$$

and split $\alpha = \alpha_{-1} + \alpha_0 + \alpha_1 + \alpha_2$ according to this decomposition. Then do the further splitting $\alpha_2 = \alpha_2' + \alpha_2''$, where $\alpha_2' = \alpha(\frac{\partial}{\partial z})dz$ and $\alpha_2'' = \overline{\alpha_2'}$. And consider the family of deformations

$$\alpha_\lambda := \lambda^{-2}\alpha_2' + \lambda^{-1}\alpha_{-1} + \alpha_0 + \lambda\alpha_1 + \lambda^2\alpha_2'', \quad \lambda \in \mathbb{C}^*.$$

Then:

Theorem 1.1 *(i) X is Lagrangian iff $\alpha''_{-1} = 0$*

(ii) X is HSL iff $\alpha''_{-1} = 0$ and, $\forall \lambda \in \mathbb{C}^*$, $d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0$.

Using this characterisation one see easily that HSL surfaces are solutions of a completely integrable system.

Note that analogous formulations work for HSL surfaces in $\mathbb{C}P^2 = SU(3)/S(U(2)\times U(1)) = U(3)/U(2)\times U(1)$, $\mathbb{C}D^2 = SU(2,1)/S(U(2)\times U(1))$, $\mathbb{C}P^1\times\mathbb{C}P^1$, $\mathbb{C}D^1\times\mathbb{C}D^1$ [3].

2 Generalizations in $\mathbb{R}^4$ (after I. Khemar [4])

Again $\mathbb{R}^4$ is endowed with its canonical Euclidean structure. We will also use an identification of $\mathbb{R}^4$ with the quaternions $\mathbb{H}$. We recall that this allows to represent rotations $R \in SO(4)$ by a pair $(p,q)\in S^3\times S^3 \subset \mathbb{H}\times\mathbb{H}$ of unit quaternions such that $\forall z\in\mathbb{H}$, $R(z) = pz\overline{q}$. In other words, denoting by $L_p : z \longmapsto pz$ and $R_{\overline{q}} : z\longmapsto z\overline{q}$, we have $R = L_pR_{\overline{q}} = R_{\overline{q}}L_p$. The pair (p,q) is unique up to sign, hence the identification $SO(4)\simeq S^3\times S^3/\{\pm\}$.

Moreover we can also precise the identification $Gr_2(\mathbb{R}^4)\simeq S^2\times S^2$. Let

$$Stiefel_2(\mathbb{H}) := \{(e_1,e_2)\in\mathbb{H}\times\mathbb{H}|\, |e_1| = |e_2| = 1, \langle e_1,e_2\rangle = 0\}.$$

Observe that $\forall(e_1,e_2)\in Stiefel_2(\mathbb{H})$, $e_2\overline{e_1}$ (resp. $\overline{e_1}e_2$) is unitary (because e_1 and e_2 are so) and imaginary (because $\langle e_1,e_2\rangle = 0$). Hence this defines two maps

$$\begin{array}{ccc} Stiefel_2(\mathbb{H}) & \longrightarrow & S^2 \\ (e_1,e_2) & \longmapsto & e_2\overline{e_1} \end{array}, \quad \begin{array}{ccc} Stiefel_2(\mathbb{H}) & \longrightarrow & S^2 \\ (e_1,e_2) & \longmapsto & \overline{e_1}e_2 \end{array}.$$

These maps factor through the natural map $P : (e_1,e_2)\longmapsto \mathrm{Span}\{e_1,e_2\}$ from $Stiefel_2(\mathbb{H})$ to the oriented Grassmannian $Gr_2(\mathbb{H})$: let

$$\begin{array}{c} \rho : Gr_2(\mathbb{H}) \longrightarrow S^2 \\ \text{s. t. } \rho\circ P(e_1,e_2) = e_2\overline{e_1} \end{array}, \quad \begin{array}{c} \sigma : Gr_2(\mathbb{H}) \longrightarrow S^2 \\ \text{s. t. } \sigma\circ P(e_1,e_2) = \overline{e_1}e_2. \end{array}$$

Then $(\rho,\sigma) : Gr_2(\mathbb{H}) \longrightarrow S^2\times S^2$ is a diffeomorphism.

2.1 Immersions of a surface in $\mathbb{H}$ with a harmonic 'left Gauss map'

Let $X : \Omega \longrightarrow \mathbb{H}$ be a conformal immersion and $\rho_X : \Omega\longrightarrow S^2$ its *left Gauss map*, i.e. $\forall z\in\Omega$, $\rho_X(z)$ is the image of $\mathrm{Span}(\frac{\partial X}{\partial x}(z), \frac{\partial X}{\partial y}(z))$ by ρ. It is characterised by

$$\frac{\partial X}{\partial y} = \rho_X\frac{\partial X}{\partial x} \quad\Longleftrightarrow\quad i\frac{\partial X}{\partial z} = \rho_X\frac{\partial X}{\partial z}.$$

(In the second equation the i on the l.h.s. is the complex structure on $\Omega \subset \mathbb{C}$, whereas the ρ_X on the r.h.s. denotes the left multiplication in $\mathbb{H}$.)

Remark: instead of viewing ρ_X as the left component of the Gauss map in $Gr_2(\mathbb{H}) \simeq S^2 \times S^2$, an alternative interpretation is that ρ_X is a map into the 'left' connected component of the manifold of compatible complex structures $\mathcal{J}_{\mathbb{H}} \simeq S^2 \cup S^2$ on $\mathbb{H}$ (cf. the work of F. Burstall).

Idea: to lift the pair (X, ρ_X) by a framing $F : \Omega \longrightarrow \mathfrak{G}$, $\mathfrak{G}$ is a subgroup of $SO(4) \ltimes \mathbb{R}^4$.
How ? We fix some constant imaginary unit vector $u \in S^2 \subset \mathrm{Im}\mathbb{H}$.

- *First method:* we lift X **and** its full Gauss map $T_X\Sigma \simeq (\rho_X, \sigma_X)$: we let (e_1, e_2) be any moving frame which is an orthonormal basis of $T_{X(z)}\Sigma$ (e.g. $e_1 = \frac{\partial X}{\partial x}/|\frac{\partial X}{\partial x}|$, $e_2 = \frac{\partial X}{\partial x}/|\frac{\partial X}{\partial y}|$) and we choose $F = (R, X)$ s.t. R satisfies:

  $$R(1) = e_1, \quad R(u) = e_2.$$

  Decompose $R = L_p R_{\overline{q}}$, then

  $$R(1) = p\overline{q}, R(u) = pu\overline{q}, \text{ so that } \quad \rho_X = e_2\overline{e_1} = pu\overline{p}.$$

  Note: In this case we must choose $\mathfrak{G} = SO(4) \ltimes \mathbb{R}^4$ (which acts transitively on $Stiefel_2(\mathbb{H})$).

- *Second method:* we lift **only** X and ρ_X. Then it means that we choose $F = (R, X)$, where $R = L_p R_{\overline{q}}$ is s.t.

  $$\rho_X = pu\overline{p}.$$

  Hence the choice of q is not relevant. In other words introducing the *(left) Hopf fibration*

  $$\begin{array}{rccc} \mathcal{H}_L^u : & SO(4) & \longrightarrow & S^2 \\ & L_p R_{\overline{q}} & \longmapsto & pu\overline{p}, \end{array}$$

  we choose the lift $F = (R, X)$ in such a way that $\mathcal{H}_L^u \circ R = \rho_X$.
  We observe that in this case one may choose $q = 1$ and assume that $R \in \{L_p | \, p \in S^3\} \simeq Spin3$, i.e. work with $\mathfrak{G} = Spin3 \ltimes \mathbb{H}$. The restriction of $\mathcal{H}_L^u$ to $Spin3$ (viewed as a subgroup of $SO(4)$) is just the Hopf fibration $\mathcal{H}^u : S^3 \longrightarrow S^2$.

Actually the second point of view is more general and leads to a simpler theory.

Now let $\tau : (R, X) \longmapsto (L_u R L_u^{-1}, -L_u X)$, a 4th order automorphism of $\mathfrak{G}$ (i.e. $\tau^4 = Id$). It induces a 4th order automorphism on its Lie algebra $\mathfrak{g}$. Let

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0^{\mathbb{C}} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2^{\mathbb{C}}$$

be its associated eigenspace decomposition. Split the Maurer–Cartan form $\alpha = F^{-1}dF$ according to this decomposition: $\alpha = \alpha_{-1} + \alpha_0 + \alpha_1 + \alpha_2$ and let

$$\beta_{\lambda^2} := \lambda^{-2}\alpha_2' + \alpha_0 + \lambda^2\alpha_2'',$$

$$\alpha_\lambda := \lambda^{-2}\alpha_2' + \lambda^{-1}\alpha_{-1} + \alpha_0 + \lambda\alpha_1 + \lambda^2\alpha_2'' = \beta_{\lambda^2} + \lambda^{-1}\alpha_{-1} + \lambda\alpha_1.$$

Then:

Lemma 2.1 *If $X : \Omega \longrightarrow \mathbb{R}^4$ is a conformal immersion and if $R : \Omega \longrightarrow SO(4)$ is an arbitrary smooth map, then*

$$\mathcal{H}_L^u \circ R = \rho_X \quad \Longleftrightarrow \quad \alpha_{-1}'' = 0.$$

In other words $F = (R, X) : \Omega \longrightarrow SO(4) \ltimes \mathbb{R}^4$ lifts (X, ρ_X) iff $\alpha_{-1}'' = 0$.

Remark: α_1 is the complex conjugate of α_{-1}, so that $\alpha_{-1}'' = 0$ iff $\alpha_1' = 0$.

Lemma 2.2 *We have:*

$$d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = d\beta_{\lambda^2} + \frac{1}{2}[\beta_{\lambda^2} \wedge \beta_{\lambda^2}] + (\lambda^{-3} - \lambda)[\alpha_2' \wedge \alpha_{-1}''] + (\lambda^3 - \lambda^{-1})[\alpha_2'' \wedge \alpha_1']. \quad (1)$$

Hence in particular, if F lifts (X, ρ_X), then $d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = d\beta_{\lambda^2} + \frac{1}{2}[\beta_{\lambda^2} \wedge \beta_{\lambda^2}]$.

In order to interpret (1) we further observe that

(i) $\mathfrak{G}^\tau$, the fixed subset of $\tau : \mathfrak{G} \longrightarrow \mathfrak{G}$, is a subgroup of $\mathfrak{G}$ with Lie algebra $\mathfrak{g}_0$

(ii) $\mathfrak{G}^{\tau^2} = \{(R, 0) \in \mathfrak{G}\}$, the fixed subset of $\tau^2 : \mathfrak{G} \longrightarrow \mathfrak{G}$, is a subgroup of $\mathfrak{G}$ with Lie algebra $\mathfrak{g}_0 \oplus \mathfrak{g}_2$,

with the inclusions

$$\mathfrak{G}^\tau \subset \mathfrak{G}^{\tau^2} \subset \mathfrak{G}.$$

Moreover $\mathfrak{G}/\mathfrak{G}^{\tau^2} \simeq \mathbb{H}$ and $\mathfrak{G}^{\tau^2}/\mathfrak{G}^\tau \simeq S^2$ and the projection map

$$\begin{array}{ccc} \mathfrak{G}^{\tau^2} & \longrightarrow & \mathfrak{G}^{\tau^2}/\mathfrak{G}^\tau \simeq S^2 \\ R \simeq (R, 0) & \longmapsto & R \bmod \mathfrak{G}^\tau \end{array}$$

coincides with the Hopf fibration $\mathcal{H}_L^u$. Hence, by applying the standard theory of harmonic maps into symmetric spaces, we deduce that:

$$d\beta_{\lambda^2} + \frac{1}{2}[\beta_{\lambda^2} \wedge \beta_{\lambda^2}] = 0 \quad \Longleftrightarrow \quad \mathcal{H}_L^u \circ R : \Omega \longrightarrow S^2 \text{ is harmonic.}$$

Putting Lemmas 2.1 and 2.2 and these observations together we conclude with the following:

Theorem 2.1 *Let* $X : \Omega \longrightarrow \mathbb{H}$ *be a conformal immersion and* $\rho_X : \Omega \longrightarrow S^2$ *its left Gauss map. Let* $F = (R, X) : \Omega \longrightarrow \mathfrak{G}$ *be any smooth map. Then*

(i) $\mathcal{H}_L^u \circ R = \rho_X$ *(i.e.* F *is a lift of* (X, ρ_X)*) iff* $\alpha''_{-1} = 0$

(ii) If so, i.e. if F *is a lift of* (X, ρ_X)*, then* ρ_X *is harmonic iff*

$$d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0.$$

2.2 Examples

2.2.1 HSL surfaces revisited

Let us introduce again the symplectic form $\omega = dx^1 \wedge dx^2 + dx^3 \wedge dx^4$. Note that $\omega = \omega_1 := \langle L_i \cdot, \cdot \rangle$. Let us introduce also $\omega_2 := \langle L_j \cdot, \cdot \rangle = dx^1 \wedge dx^3 + dx^4 \wedge dx^2$ and $\omega_3 := \langle L_k \cdot, \cdot \rangle = dx^1 \wedge dx^4 + dx^2 \wedge dx^4$. Then

$$e_2\overline{e_1} = \rho(e_1, e_2) = i\omega_1(e_1, e_2) + j\omega_2(e_1, e_2) + k\omega_3(e_1, e_2).$$

So X is a conformal *Lagrangian* immersion iff $X^*\omega_1 = 0$, i.e. iff ρ_X takes values in

$$S^1 = \{j \cos\beta + k \sin\beta = e^{i\beta} j |\, \beta \in \mathbb{R}\}.$$

Hence a lift of (X, ρ_X) is characterized by

$$pu\overline{p} = \mathcal{H}_L^u \circ R = \rho_X = e^{i\beta} j.$$

A convenient choice for u is to assume that $u \perp i$, e.g. $u = j$. In that case

$$\{p \in S^3 |\, pu\overline{p} = e^{i\beta} j\} = \{e^{i\beta/2} e^{j\theta} |\, \theta \in \mathbb{R}\}$$

and the simplest choices are $p = \pm e^{i\beta/2}$.

With this choice:

- if we start with the group $\mathfrak{G} = SO(4) \ltimes \mathbb{R}^4$, our lift satisfies $R = L_{e^{i\beta/2}} R_{\overline{q}}$, i.e. we can reduce $SO(4) \ltimes \mathbb{R}^4$ to $U(2) \ltimes \mathbb{C}^2$
- if we start with the group $\mathfrak{G} = Spin3 \ltimes \mathbb{H}$, our lift satisfies $R = L_{e^{i\beta/2}}$, i.e. we can reduce $Spin3 \ltimes \mathbb{R}^4$ to $U(1) \ltimes \mathbb{C}^2$ (cf. spinor lifts, related to the Konopelchenko–Taimanov representation formula).

2.2.2 Constant mean curvature surfaces in $\mathbb{R}^3$

Consider an immersed surface Σ in $\mathbb{H}$ with a harmonic left Gauss map. If we assume further that this surface is contained in Im$\mathbb{H}$, then any orthonormal basis (e_1, e_2) of $T_{X(z)}\Sigma$ is composed of imaginary vectors. Hence

$$\rho_X = e_2\overline{e_1} = -\overline{e_1} e_2 = -\sigma_X,$$

so that ρ_X is harmonic iff σ_X is so. Actually ρ_X is nothing but the Gauss map of Σ in Im$\mathbb{H} \simeq \mathbb{R}^3$. Hence by Ruh–Vilms theorem we know that Σ is a *constant mean curvature surface* in $\mathbb{R}^3$. Conversely any constant mean curvature surface in $\mathbb{R}^3$ arises that way.

2.3 Other generalizations in dimension 4

This theory can be generalized to surfaces in S^4 or $\mathbb{C}P^2$: then (X, ρ_X) is replaced by a lift of the immersion X in the four dimensional manifold into the twistor bundle of complex structures. The condition of ρ_X being harmonic is replaced by the fact this lift is vertically harmonic (the fiber being the set of (left) compatible complex structures, diffeomorphic to S^2). This follows from independant works by F. Burstall and I. Khemar.

3 A generalization for surfaces in $\mathbb{R}^8$ (I. Khemar [4])

The following theory is based on the identification of $\mathbb{R}^8$ with octonions $\mathbb{O}$. Again the map

$$\begin{array}{ccc} Stiefel_2(\mathbb{O}) & \longrightarrow & S^6 \\ (e_1, e_2) & \longmapsto & e_2\overline{e_1}, \end{array}$$

where $S^6 \in \mathrm{Im}\mathbb{O} \subset \mathbb{O}$, can be factorized through the map $P : Stiefel_2(\mathbb{O}) \longrightarrow Gr_2(\mathbb{O})$, $(e_1, e_2) \longmapsto \mathrm{Span}\{e_1, e_2\}$ by introducing

$$\begin{array}{c} \rho : Gr_2(\mathbb{O}) \longrightarrow S^6 \\ \text{s.t. } \rho \circ P(e_1, e_2) = e_2\overline{e_1}. \end{array}$$

Let Σ be an immersed surface in $\mathbb{O}$ we say that Σ *is ρ-harmonic iff the composition of the Gauss map $\Sigma \longrightarrow Gr_2(\mathbb{O})$ with ρ is harmonic.*

This theory is completely similar with the theory of surfaces in quaternions $\mathbb{H}$ which used the group $\mathfrak{G} = Spin3 \ltimes \mathbb{H}$, where $Spin3$ can be seen as the subgroup of $SO(4)$ generated by L_i, L_j and L_k and the induced representation of $Spin3$ was the spinor representation $\mathbb{H}$. Here we will use $\mathfrak{G} = Spin7 \ltimes \mathbb{O}$, where $Spin7$ can be identified with the subgroup of $SO(8)$ generated by $\{L_v |\, v \in S^6 \subset \mathrm{Im}\mathbb{O}\}$ and the induced representation on $\mathbb{R}^8$ coincides with the spinor representation of $Spin7$ on $\mathbb{O}$. A difference however is that $Spin7$ is "bigger" than $Spin3$ and in particular acts transitively on $Stiefel_2(\mathbb{O})$ (with isotropy $SU(3)$) and $Gr_2(\mathbb{O})$ (with isotropy G_2), whereas $Spin3$ do not act transitively on $Gr_2(\mathbb{H})$. After fixing an imaginary unit octonion $u \in \mathbb{O}$, a 'Hopf' fibration

$$\begin{array}{cccc} \mathcal{H}^u : & Spin7 & \longrightarrow & S^6 \\ & p & \longmapsto & \mathcal{H}^u(p), \text{ s.t. } pL_u p^{-1} = L_{\mathcal{H}^u(p)} \end{array}$$

can be defined.

Now let $X : \mathbb{C} \supset \Omega \longrightarrow \mathbb{O}$ be a conformal immersion and denote $\rho_X := \rho \circ T_X\Sigma$ the composition of the Gauss map $T_X\Sigma$ of X with ρ. After having fixed $u \in S^6 \subset \mathbb{O}$ we let

$$F = \begin{pmatrix} R & X \\ 0 & 1 \end{pmatrix} \simeq (R, X) : \Omega \longrightarrow Spin7 \ltimes \mathbb{H},$$

be a smooth map. We say that F lifts (X, ρ_X) iff $\mathcal{H}^u \circ R = \rho_X$. Using the 4th order automorphism $\tau : \mathfrak{G} \longrightarrow \mathfrak{G}$ defined by

$$\tau(R, X) = (L_u R L_u^{-1}, -L_u X),$$

we can characterized among all maps $F = (R, X)$ those which lift ρ_X by the condition $\alpha''_{-1} = 0$ (after a decomposition of the Maurer–Cartan form $\alpha := F^{-1}dF$ along the eigenspaces of the action of τ on the Lie algebra $\mathfrak{g}$ of $\mathfrak{G}$). Then the ρ-harmonic immersions satisfy a zero curvature equation $d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0$ similar to the previous case.

Again ρ_X can be interpreted as a map into the manifold $\mathcal{J}_{\mathbb{O}}$ of compatible complex structures on $\mathbb{O}$, because of the relation $\rho_X \frac{\partial X}{\partial z} = i\frac{\partial X}{\partial z}$. However the embedding $S^6 \subset \mathcal{J}_{\mathbb{O}}$ is much less clear than the inclusion $S^2 \subset \mathcal{J}_{\mathbb{H}}$ that we used previously: we recall indeed that $\mathcal{J}_{\mathbb{H}} \simeq S^2_L \cup S^2_R$ and hence that our S^2 was just the (left) connected component of $\mathcal{J}_{\mathbb{H}}$. However $\mathcal{J}_{\mathbb{O}} \simeq SO(8)/U(4)$ is 12 dimensional, so that our S^6 is now a particular submanifold of $\mathcal{J}_{\mathbb{O}}$. Hence a twistor intepretation of the theory in $\mathbb{O}$ seems less clear.

4 Towards a supersymmetric interpretation

Observation : the coefficients of α_{-1} and α_1 actually behave like spinors (they turn half less than those of α_2 when λ run over S^1 and they satisfy a kind of Dirac equation). This motivates the following results by I. Khemar [5].

4.1 Superharmonic maps into a symmetric space

For simplicity we restrict ourself to maps into the sphere $S^n \subset \mathbb{R}^{n+1}$. It can be seen as a system of PDE's on a map $u : \Omega \longrightarrow S^n$ (where $\Omega \subset \mathbb{C}$) and *odd* sections ψ_1, ψ_2 of u^*TS^n. This system is

$$\begin{cases} \nabla_{\overline{z}} \dfrac{\partial u}{\partial z} = \dfrac{1}{4}\left(\psi\langle\psi, \dfrac{\partial u}{\partial \overline{z}}\rangle - \overline{\psi}\langle\overline{\psi}, \dfrac{\partial u}{\partial z}\rangle\right) \\ \nabla_{\overline{z}}\psi = \dfrac{1}{4}\langle\overline{\psi}, \psi\rangle\overline{\psi}, \end{cases} \quad (2)$$

where $\psi = \psi_1 - i\psi_2$. By "odd" we mean that the components ψ_1 and ψ_2 are anticommuting (Grassmann) variables. An alternative elegant reformulation of this system can be obtained by adding the extra field $F : \Omega \longrightarrow \mathbb{R}^{n+1}$, which satisfies the 0th order PDE's

$$F = \frac{1}{2i}\langle\psi, \overline{\psi}\rangle u \quad (3)$$

and by setting

$$\Phi := u + \theta^1\psi_1 + \theta^2\psi_2 + \theta^1\theta^2 F,$$

where θ^1 and θ^2 are anticommuting coordinates, so that $(x, y, \theta^1, \theta^2)$ forms a complete system of coordinates on the *superplane* $\mathbb{R}^{2|2}$. Then (2) and (3) are equivalent to

$$\overline{D}D\Phi + \langle\overline{D}\Phi, D\Phi\rangle\Phi = 0, \quad (4)$$

where $D = \frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial z}$, $\overline{D} = \frac{\partial}{\partial\overline{\theta}} - \overline{\theta}\frac{\partial}{\partial\overline{z}}$.

Actually, from (2) and (3) to (4), we have used the fact that u, ψ_1, ψ_2 and F are the components (supermultiplet) of a single map Φ from $\mathbb{R}^{2|2}$ to $S^n \subset \mathbb{R}^{n+1}$, which satisfies the superharmonic map equation (4).

Now we lift Φ to a framing supermap $\mathcal{F} : \mathbb{R}^{2|2} \longrightarrow SO(n+1)$ such that the composition of $\mathcal{F}$ with the projection $SO(n+1) \longrightarrow SO(n+1)/SO(n) \simeq S^n$ is Φ. Set $\alpha := \mathcal{F}^{-1}d\mathcal{F}$ and decompose $\alpha = \alpha_0 + \alpha_1$, according to the splitting of the Lie algebra $so(n+1)$ by the Cartan involution.

Before giving a characterization of the superharmonic equation, it is useful to present a technical result concerning the exterior calculus of 1-forms on $\mathbb{R}^{2|2}$.

Lemma 4.1 *For a 1-form α on $\mathbb{R}^{2|2}$ with coefficients in a Lie algegra $\mathfrak{g}$, we have the equivalence*

$$d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0 \quad \Longleftrightarrow \quad \overline{D}\alpha(D) + D\alpha(\overline{D}) + [\alpha(\overline{D}), \alpha(D)] = 0.$$

Remark: $\Lambda^1(\mathbb{R}^{2|2})^*$ is spanned by $(d\theta, d\overline{\theta}, dz+(d\theta)\theta, d\overline{z}+(d\overline{\theta})\overline{\theta})$, the dual basis of $(D, \overline{D}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}})$. Hence in particular $\Lambda^2(\mathbb{R}^{2|2})^*$ is 6 dimensional. So the expansion of the l.h.s. of $d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0$ leads to 6 equations which are a priori independant. The content of this lemma is that these 6 terms vanish as soon as one of these 6 coefficients (namely the coefficient of $d\theta \wedge d\overline{\theta}$) vanishes.

Now the supermap $\mathcal{F}$ is superharmonic iff

$$\overline{D}\alpha_1(D) + [\alpha_0(\overline{D}), \alpha_1(D)] = 0.$$

We hence deduce:

Theorem 4.1 *$\mathcal{F}$ is superharmonic iff*

$$\forall\lambda \in \mathbb{C}^*, \quad \overline{D}\alpha(D)_\lambda + D\alpha(\overline{D})_\lambda + [\alpha(\overline{D})_\lambda, \alpha(D)_\lambda] = 0,$$

where $\alpha(D)_\lambda := \alpha_0(D) + \lambda^{-1}\alpha_1(D)$ and $\alpha(\overline{D})_\lambda := \alpha_0(\overline{D}) + \lambda\alpha_1(\overline{D})$.

It results that this problem has the structure of a completely integrable system (F. O'Dea, I. Khemar). In particular the DPW algorithm for harmonic maps works.

The DPW potential is a $\Lambda\mathfrak{g}_\tau^{\mathbb{C}}$-valued holomorphic 1-form μ on $\mathbb{R}^{2|2}$ s.t.

$$\mu(D) = \mu_0(D) + \theta\mu_\theta(D) = \lambda^{-1}(\cdot) + \lambda^0(\cdot) + \cdots$$

One integrates the equation

$$Dg = g\mu(D)$$

to get a holomorphic map $g = g_0 + \theta g_\theta : \mathbb{R}^{2|2} \longrightarrow \Lambda\mathfrak{G}_\tau^{\mathbb{C}}$. This implies in particular that

$$g_0^{-1}\frac{\partial g_0}{\partial z} = -\left((\mu_0(D))^2 + \mu_\theta(D)\right) = \lambda^{-2}(\cdot) + \lambda^{-1}(\cdot) + \lambda^0(\cdot) + \cdots$$

Similarly, if $\mathcal{F} = \mathcal{F}_0 + \theta\mathcal{F}_\theta + \bar\theta\mathcal{F}_{\bar\theta} + \theta\bar\theta\mathcal{F}_{\theta\bar\theta}$, it turns out that $\mathcal{F}_0^{-1}d\mathcal{F}_0 = \lambda^{-2}(\cdot) + \lambda^{-1}(\cdot) + \lambda^0(\cdot) + \lambda(\dot{}) + \lambda^2(\cdot)$. Hence we recover (for $\mathcal{F}_0$) something similar to a second order elliptic integrable system.

4.2 Superprimitive maps [5]

More precisely we can recover a second order elliptic integrable system close to the HSL surface theory in $\mathbb{R}^4$ by looking at *superprimitive maps* from $\mathbb{R}^{2|2}$ to the 4-symmetric space $SU(3)/SU(2)$: if $\Phi : \mathbb{R}^{2|2} \longrightarrow SU(3)/SU(2)$ is a superprimitive map then the first component u in the decomposition $\Phi = u + \theta^1\psi_1 + \theta^2\psi_2 + \theta^1\theta^2 F$ is a conformal HSL immersion (with the restriction that the Lagrangian angle β is equal to a *real* constant plus a harmonic non constant *nilpotent* function).

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