Asymptotically de Sitter Einstein-Weyl geometries in 2+1 dimensions

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Theme: Holomorphic discs & problems in differential geometry

Work in progress with Claude LeBrun (& some with David Calderbank).

Some motivation from integral geometry

The Radon transform:

Given a function $f(\lambda, \mu)$ on $\mathscr{T}_{\mathbb{R}} = \mathbb{R}^2$, we can integrate along the lines

$$L_{(x_0,x_1)} = \{(\lambda, \mu) | \mu = x_0 + \lambda x_1\}$$

to obtain

$$f(\lambda, \mu) \longrightarrow \hat{f}(x_0, x_1) = \int_{L(x_0,x_1)} f(\lambda, x_0 + \lambda x_1) \mathrm{d}\lambda$$

this is an isomorphism (under suitable analytic assumptions).

Question: what if we replace lines by other curves?

(0) Parabolas: in $\mathbb{R}^2$ are given by

$$C_{(x_0,x_1,x_2)} = \{(\lambda, \mu) | \mu = x_0 + \lambda x_1 + \lambda^2 x_2\}\,.$$

The space of parabolas is $\mathscr{M} = \mathbb{R}^3$ and the transform

$$f(\lambda, \mu) \longrightarrow \hat{f}(x_0, x_1, x_2) = \int_{C_{(x_0,x_1,x_2)}} f(\lambda, x_0 + \lambda x_1 + \lambda^2 x_2) \mathrm{d}\lambda \quad \in C^\infty(\mathscr{M})$$

can no longer be an isomorphism, but:
the range can be characterised (under suitable analytic hypotheses) by

$$\left(\frac{\partial^2}{\partial x_0 \partial x_2} - \frac{\partial^2}{\partial x_1^2} \right) \hat{f} = 0$$

(+) Circles:

Consider circles C_x in $\mathscr{T}_{\mathbb{R}} = S^2$. The space of oriented circles is

$$\mathscr{M}_{\mathrm{dS}} = S^2 \times \mathbb{R} = \{\text{axis direction}, t = \cot\psi).$$

For $f \in C^\infty(S^2)$, $x \in \mathscr{M}_{\mathrm{dS}}$, define $f \longrightarrow \hat{f}(x) = \int_{C_x} f \in C^\infty(\mathscr{M})$.

This time we have $\Box_{g_{\mathrm{dS}}} \hat{f} = 0$ where

$$g_{\mathrm{dS}} = \frac{\mathrm{d}t^2}{(1+t^2)} - (1+t^2) ds^2_{S^2}$$

is the $2+1$-dimensional de-Sitter metric.

(-) Hyperbolae:

Let $\mathbb{R}^2 \subset \mathscr{T}_{\mathbb{R}} = \mathbb{RP}^1 \times \mathbb{RP}^1$, the quadric of signature $(2,2)$ in $\mathbb{RP}^3$.

A hyperbola $C_x \subset \mathscr{T}_{\mathbb{R}} \subset \mathbb{RP}^3$ is the intersection of a plane with $\mathscr{T}_{\mathbb{R}}$.

The space of oriented hyperbolae is $\mathscr{M}_{\mathrm{adS}} = S^1 \times \mathbb{R}^2$ (solid torus).

For $f \in C^\infty(S^1 \times S^1)$, $x \in \mathscr{M}_{\mathrm{adS}}$, define $f \longrightarrow \hat{f}(x) = \int_{C_x} f$.

Then $\hat{f} \in C^\infty(\mathscr{M}_{\mathrm{adS}})$ such that $\Box_{g_{\mathrm{adS}}} \hat{f} = 0$ where

$$g_{\mathrm{adS}} = (1 + r^2)\mathrm{d}\theta^2 - \frac{\mathrm{d}r^2}{(1 + r^2)} - r^2 d\phi^2 \,,$$

is the $2 + 1$-dimensional anti de-Sitter metric on $\mathscr{M}_{\mathrm{adS}}$.
(Here $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, $(r, \phi) =$ polar coords on $\mathbb{R}^2$—note periodicity of time).

Questions:

- When does integration over a three-parameter family of curves $C_x \subset \mathscr{T}^2_{\mathbb{R}}$, $x \in \mathscr{M}^3$ give rise to solutions to a wave equation?

- What geometries arise?

We have correspondence space $\mathscr{F}_{\mathbb{R}} = \{(x,(\lambda,\mu)) \in \mathscr{M} \times \mathscr{T}_{\mathbb{R}} | z \in C_x\}$ with double fibration

$$\begin{array}{ccccc} & & \mathscr{F}_{\mathbb{R}} & & \\ p & \swarrow & & \searrow & q \\ \mathscr{M} & & & & \mathscr{T}_{\mathbb{R}} \end{array}$$

The curves through $(\lambda,\mu) \in \mathscr{T}_{\mathbb{R}}$ form 2-surface $\Sigma_{(\lambda,\mu)} = p(q^{-1}(\lambda,\mu)) \subset \mathscr{M}$.

by considering δ-functions on $\mathscr{T}_{\mathbb{R}}$, the $\Sigma_{(\lambda,\mu)}$ must be characteristic and $\exists$ a compatible connection for which they are totally geodesic.

Theorem 1. [Cartan 1941] *Let $\mathscr{M}$ arise as above such that the characteristics are compatible with a Lorentzian metric, then $\mathscr{M}$ is an Einstein-Weyl space.*

Definition 1. *An Einstein-Weyl space in $2+1$ dimensions is a three manifold $\mathscr{M}$ equipped with*

- *a conformal class of Lorentzian metrics $[g]$,*
- *a torsion-free affine connection $\nabla : \Gamma(T\mathscr{M}) \to \Omega^1 \otimes T\mathscr{M}$*

such that $\nabla[g] = 0$ and $Sym_0 Ricci(\nabla) = 0$.

- Cartan shows that the equations determine evolution from initial data of four free functions of 2 variables.

- If $\exists g \in [g]$ such that $\nabla g = 0$, then the metric is flat, dS or adS.

- Einstein-Weyl equations $\Leftrightarrow$ integrability of the 2-planes $\Sigma_{(\lambda,\mu)}$ and hence $\Leftrightarrow \exists \mathscr{T}_{\mathbb{R}}$ (Lax pair description).
$\rightsquigarrow$ the equations are an 'integrable system'.

- The geometry is the most general 3-dimensional geometry on which the Bogomolny equations $F_A = *D_A\Phi$ on a connection D_A on a bundle $E \to \mathscr{M}$ plus Higgs field $\Phi \in \mathrm{End}(E)$ are an integrable system.
$\rightsquigarrow$ notion of an integrable background geometry.

- This geometry is the non-linear part of the generic symmetry reduction from anti-self-dual conformal structures in four dimensions (and hence hyper-complex or hyper-kahler spaces, scalar-flat Kahler manifolds, but all in split signature).

Symmetry Reductions include:

1. $\mathrm{SU}(\infty)$ Toda equations:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 \mathrm{e}^u}{\partial t^2}\,.$$

   when there is a geodesic shear free and twist-free congruence of time-like geodesics.

2. The dispersionless KP equations

$$u_{tx} - (uu_x)_x = u_{yy}\,.$$

   when there is a constant weighted (null) vector field.

3. The spinor vortex equations for a metric g and spinor ψ in 2-dimensions

$$\not{D}\psi = \frac{3}{2}\psi\,, \qquad R = |\psi|^2 - 1\,.$$

This is the generic symmetry reduction.

Families of curves and holomorphic discs

A 3-parameter family of curves in 2-dim $\leftrightarrow$ a free function of four variables:

$$\mu = f(\lambda, x_0, x_1, x_2),$$

whereas Einstein-Weyl spaces depend on just functions of 2 variables.

Question: Can we characterise families of curves for Einstein-Weyl spaces?

Complex analysis: In the de Sitter case, $\mathscr{T}_{\mathbb{R}} = S^2$; we must understand this as the antiholomorphic diagonal inside $\mathscr{T} = \mathbb{CP}^1 \times \mathbb{CP}^1$.

Holomorphic discs: Oriented circles in $\mathscr{T}_{\mathbb{R}}$ can be characterized as those oriented closed curves in $\mathscr{T}_{\mathbb{R}}$ that bound holomorphic discs $D \subset \mathscr{T}$ with $\partial D \subset \mathscr{T}_{\mathbb{R}}$ in appropriate topological class.

Thus, $\mathscr{M}_{dS}$ = moduli space of holomorphic discs $D \subset \mathscr{T}$ with $\partial D \subset \mathscr{T}_{\mathbb{R}}$.

Theorem 2. *Let* $(\mathscr{M}^3, [g], \nabla)$ *be an Einstein-Weyl space with* $\mathscr{M} = S^2 \times \mathbb{R}$ *that is asymptotically de Sitter and is oriented and time oriented compatibly with the asymptotic structure.*

Let $\mathscr{T}_{\mathbb{R}} = \{$*totally geodesic null 2-planes in* $\mathscr{M}\}$*.*

Then $\mathscr{T}_{\mathbb{R}}$ *is* S^2 *and admits a canonical embedding* $\mathscr{T}_{\mathbb{R}} \hookrightarrow \mathscr{T} = \mathbb{CP}^1 \times \mathbb{CP}^1$*.*

$\mathscr{M}$ *can be reconstructed as the moduli space of embedded holomorphic discs* D_x *in* $\mathbb{CP}^1 \times \mathbb{CP}^1$ *such that* $\partial D_x \subset \mathscr{T}_{\mathbb{R}}$*.*

Theorem 3. *There is a 1:1 correspondence between oriented and time oriented asymptotically anti de Sitter Einstein-Weyl spaces on* $S^2 \times \mathbb{R}$ *and (small) deformations of the embedding of the anti-holomorphic diagonal in* $\mathbb{CP}^1 \times \mathbb{CP}^1$*.*

Definition 2. $(\mathscr{M}, [g], \nabla)$ *is asymptotically de Sitter if*

- $\exists$ *conformal compactification* $(\tilde{\mathscr{M}}, \tilde{g}, \tilde{\nabla})$ *with* $\tilde{\mathscr{M}} = S^2 \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ *and* $\mathscr{M} \simeq S^2 \times (-\frac{\pi}{2}, \frac{\pi}{2}) \subset \tilde{\mathscr{M}}$ *(cf. Penrose),*

- $\tilde{g}$ *smooth on* $\tilde{\mathscr{M}}$ *and* $\tilde{g} \in [g]$ *on* $\mathscr{M}$*.*

- $\tilde{\nabla} = \nabla$ *on* $\mathscr{M}$*, and* $\tilde{\nabla}\tilde{g} = \nu \tilde{g}$ *where* ν *has a simple pole in* τ *at* $\pm\frac{\pi}{2}$*, where* τ *is coordinate on* $[-\frac{\pi}{2}, \frac{\pi}{2}]$ *factor (cf.* $t = \tan\tau$ *in de Sitter case).*

Proof of theorem:

- Let $\mathscr{F}_{\mathbb{R}} \subset \mathbb{P}T^*\mathscr{M} \to \mathscr{M}$ be the S^1-bundle of real null co-vectors; let $\mathscr{F} \to \mathscr{M}$ be the $\mathbb{CP}^1$ bundle of complex null co-vectors.

- $\mathscr{F}$ is divided into two parts $\mathscr{F}^{\pm}$ by $\mathscr{F}_{\mathbb{R}}$; e.g., choose $\mathscr{F}^{+}$ to be those null vectors that induce a spatial complex structure agreeing with the spatial orientation.

- Define a 3-dim complex distribution $\mathscr{D}$ on $\mathscr{F}^{+}$ by

$$\mathscr{D}_{(n,x)} = \{\ker n \cap Hor_{\nabla}, \partial/\partial\bar{n}\}$$

where $x \in \mathscr{M}$, $n \in \mathscr{F}^{+}|_x$, $\partial/\partial\bar{n}$ is the d-bar operator in the direction of the $\mathbb{CP}^1$ fibres of $\mathscr{F}$.

- The Einstein-Weyl equations $\Leftrightarrow \mathscr{D}$ is Frobenius integrable.

- $\mathscr{D}$ has dim 3, and $\mathscr{D} \cap \bar{\mathscr{D}}$ has dim 2 on $\mathscr{F}_{\mathbb{R}}$ but 1 on $\mathscr{F} - \mathscr{F}_{\mathbb{R}}$.

- Define $\mathscr{T} = \mathscr{F}^{+}/\mathscr{D} \cap \bar{\mathscr{D}}$; $\mathscr{D}$ descends to endow $\mathscr{T}$ with an integrable complex structure.

- With given assumptions, $\mathscr{T}$ is topologically $S^2 \times S^2$. By checking asymptotics, it can be seen to be $\mathbb{CP}^1 \times \mathbb{CP}^1$ as a complex manifold.

- $\mathscr{T}_{\mathbb{R}} = \mathscr{F}_{\mathbb{R}}/\mathscr{D} \cap \bar{\mathscr{D}}$ is a 2-dim totally real submanifold of $\mathscr{T}$.

- Each $x \in \mathscr{M} \leftrightarrow$ a holomorphic disc $D_x = \mathscr{F}^{+}|_x$ with $\partial D_x = \mathscr{F}_{\mathbb{R}}|_x$. This projects to a holomorphic disc $D_x \subset \mathscr{T}$ with $\partial D_x \subset \mathscr{T}_{\mathbb{R}}$.

Proof that $(\mathscr{M}^3, [g], \nabla)$ can be reconstructed from the embedding $\mathscr{T}_\mathbb{R} \hookrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$:

The task of finding a holomorphic disc $D \subset \mathscr{T}$ with $\partial D \subset \mathscr{T}_\mathbb{R}$ is an elliptic boundary value problem with Fredholm linearization.

The moduli space of such discs in the appropriate topological class is necessarily 3-dim, and gives $\mathscr{M}$.

Each point $z \in \mathscr{T}_\mathbb{R}$ corresponds to a two-surface Σ_z in $\mathscr{M}$ where, for $x \in \mathscr{M}$, $x \in \Sigma_z \Leftrightarrow z \in \partial D_x$.

$\exists$! Einstein-Weyl structure on $\mathscr{M}$ for which these two-surfaces are totally geodesic null surfaces. $\square$

Remark: Note that points at infinity correspond to the limiting case where D is a $\mathbb{CP}^1$ and intersects $\mathscr{T}_\mathbb{R}$ in a point.

Proof: that, arbitrary small deformations of $\mathscr{T}_{\mathbb{R}}$ correspond to asymptotically de Sitter Einstein-Weyl spaces.

Such elliptic boundary value problems are, via the implicit function theorem, stable under small deformations.

Thus, the reconstruction can be performed when $\mathscr{T}_{\mathbb{R}} \hookrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ is any small deformation of the standard embedding of the anti-holomorphic diagonal S^2 in $\mathbb{CP}^1 \times \mathbb{CP}^1$, i.e., $\exists$ a 3-parameter family $\mathscr{M}^3$ of holomorphic discs, $D_x \hookrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ with $\partial D_x \subset \mathscr{T}_{\mathbb{R}}$.

The existence of an Einstein-Weyl structure on $\mathscr{M}$ for which the Σ_z are totally geodesic null surfaces is no longer trivial, but follows by standard arguments. $\square$

Other cases & reductions

Asymptotically anti-de Sitter case: $\mathscr{T}_{\mathbb{R}} \simeq S^1 \times S^1$ is a small deformation of $\mathbb{RP}^1 \times \mathbb{RP}^1 \subset \mathbb{CP}^1 \times \mathbb{CP}^1$.

Asymptotically flat case: $\mathscr{T} \simeq$ Hirzebruch surface $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2)$, $\mathscr{T}_{\mathbb{R}} \simeq S^1 \times S^1$.

Theorem 4. *Solutions reduce to $\mathrm{SU}(\infty)$ Toda if $\mathscr{T}_{\mathbb{R}}$ is Lagrangian for the symplectic structure $\Im\Omega$, Ω is a meromorphic 2-form with two double poles. With one quadruple pole on each conic, the reduction is DKP.*

Theorem 5. *A lorentizian spinor vortex geometry arises when $\exists p : \mathscr{T} \to \mathbb{CP}^1$ such that $\mathscr{T}_{\mathbb{R}} \simeq p^{-1}$(an S^1 in $\mathbb{CP}^1$). The holomorphic discs are p^{-1} of Riemann maps $D \to \mathbb{CP}^1$ with ∂D on subintervals of the S^1. The Lax pair operators are the Loewner differential equations for the Riemann maps.*

Further twistor constructions based on holomorphic discs

LeBrun & M, math.DG/0211021, J. Diff. Geom. **61**, 2002:

$$\left\{\begin{matrix}\text{Zoll projective structures} \\ \text{on } S^2\end{matrix}\right\} \stackrel{1:1}{\longleftrightarrow} \left\{\begin{matrix}\text{Deformations of embedding } \mathbb{RP}^2 \subset \\ \mathbb{CP}^2\end{matrix}\right\}$$

LeBrun & M, math.DG/0504582, Duke:

$$\left\{\begin{matrix}\text{Self-dual conformal} \\ \text{structures on } S^2 \times S^2\end{matrix}\right\} \stackrel{1:1}{\longleftrightarrow} \left\{\begin{matrix}\text{Deformations of embedding } \mathbb{RP}^3 \subset \\ \mathbb{CP}^3\end{matrix}\right\}$$

M, math-ph/0505039, Crelle:

$$\left\{\begin{matrix}\text{global self-dual } \mathrm{U}(n) \text{ Yang-Mills fields} \\ \text{in split signature on } S^2 \times S^2 \\ (= 2 \text{ copies of } \mathbb{R}^{2,2}),\end{matrix}\right\} \stackrel{1:1}{\longleftrightarrow} \left\{\begin{matrix}\text{Hol. Vector bundle} \\ E \to \mathbb{CP}^3 \text{ \& hermitian metric} \\ H \text{ on } E|_{\mathbb{RP}^3}\end{matrix}\right\}.$$