Isotropic minimal surfaces and holomorphic curves in flat tori

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Well-known fact:

Complex submanifolds of Kähler manifolds minimize volume in their homology class and, in particular, are minimal.

Questions:

Can a holomorphic curve be deformed so as to keep it minimal but not holomorphic?

If yes, is stability of the holomorphic curve as a minimal surface preserved by the deformation?

Theorem (C. Arezzo, -)

Let $f\colon (\Sigma_\gamma, \mu) \to \mathbb{C}^n/\Lambda$ be a full holomorphic immersion of a Riemann surface of genus $\gamma \geqslant 4$. If

$$\frac{1}{2}n(n+1) > 3\gamma - 3 \geqslant 3n - 3$$

then f can be deformed as a conformal minimal immersion $f_t\colon (\Sigma, \mu) \to \mathbb{R}^{2n}/\Lambda_t$ into a flat torus $\mathbb{R}^{2n}/\Lambda_t$ which is *not* holomorphic with respect to *any* orthogonal complex structure on the torus.

If the Riemann surface (Σ_γ, μ) is hyperelliptic, then f_t will be unstable for $t \neq 0$. (An old result of - .)

If (Σ_γ, μ) is not hyperelliptic and f is the Abel-Jacobi embedding, then f_t will be stable (and nonholomorphic).

Key ingredients of proof:

(i) Weierstrass representation of conformal minimal immersions into a flat torus.

(ii) Characterisation of Weierstrass representation of holomorphic curves.

(iii) Dimension of the kernel of the map from the symmetric square of the space of holomorphic differentials in the Weierstrass representation to the space of holomorphic quadratic differentials.

(iv) Complex version of the second variation formula and Birkhoff-Grothendieck decomposition of holomorphic vector bundles over the two-sphere.

(v) An exact sequence argument to show that when the Abel-Jacobi embedding of a nonhyperelliptic Riemann surface of genus $\geqslant 4$ is viewed as a minimal immersion then its Jacobi fields are just the translations.

What about deformations of a holomorphic curve in a flat torus of complex dimension 2 or 3?

Theorem (- , E. Nedita)

Consider the following smoothly varying 1-parameter families:

- conformal structures μ_t on a surface Σ;
- lattices Λ_t of $\mathbb{R}^4$;
- conformal minimal immersions $f_t\colon (\Sigma, \mu_t) \to \mathbb{R}^4/\Lambda_t$.

If f_0 is holomorphic with respect to some orthogonal complex structure J_0 then, for each t, there exists an orthogonal complex structure J_t with respect to which f_t is holomorphic.

The proof is an easy consequence of the following fact:

Let K and $K^{\perp}$ denote respectively the Gauss curvature and the curvature of the normal bundle of a minimal immersion $f\colon \Sigma \longrightarrow \mathbb{R}^4$. Then

$$|K^{\perp}| \leqslant (-K)$$

with equality if, and only if, f is holomorphic with respect to an orthogonal complex structure on $\mathbb{R}^4$.

This proof also works for:

- a 1-parameter family of conformal minimal immersions into a hyperkähler 4-manifold. (cf. work with Wolfson on the (elaborate) construction of stable nonholomorphic two-spheres in some K-3 surfaces.)
- a 1-parameter family of conformal minimal immersions of finite total curvature in $\mathbb{R}^4$. (cf. old theorem of - .)

Alternative proof:

Given an immersion $f\colon \Sigma \to \mathbb{R}^4$ define J_f on $f^*(T\mathbb{R}^4)$ by anticlockwise rotation by 90° in the tangent plane and anticlockwise rotation by 90° in the normal plane.

The set of orthogonal complex structures on $\mathbb{R}^4$ compatible with a given orientation is a two-sphere. If f is minimal then $J_f\colon \Sigma \to S^2$ is holomorphic.

$\dots$.

What about higher dimensions?

Recall Calabi's notion of isotropicity for a minimal immersion $f\colon \Sigma \longrightarrow \mathbb{R}^n$:
f is isotropic to order l if

$$(\partial_z^j f \cdot \partial_z^k f) = 0 \qquad \forall\, j, k \in \{1, \ldots, l\}.$$

Thus isotropicity to order 1 is equivalent to conformality.

Proposition

A minimal immersion $f\colon \Sigma \longrightarrow \mathbb{R}^{2n}$ is holomorphic with respect to some orthogonal complex structure on $\mathbb{R}^{2n}$ if, and only if, f is isotropic to order n.

Theorem (- , E. Nedita)

Consider the following smoothly varying 1-parameter families:

- conformal structures μ_t on a surface Σ;
- lattices Λ_t of $\mathbb{R}^{2n}$;
- conformal minimal immersions $f_t\colon (\Sigma, \mu_t) \to \mathbb{R}^{2n}/\Lambda_t$ all of which are isotropic to order $n-1$.

If f_0 is holomorphic with respect to some orthogonal complex structure J_0 then, for each t, there exists an orthogonal complex structure J_t with respect to which f_t is holomorphic.

A similar statement holds for a family of $(n-1)$-isotropic minimal surfaces of finite total curvature in $\mathbb{R}^{2n}$.

Lemma

Let $f\colon \Sigma \longrightarrow \mathbb{R}^{2n}$ be minimal and isotropic to order $n-1$. Define

$$J_f\colon \Sigma \longrightarrow SO(2n)/U(n)$$

by 90° anticlockwise rotation in the osculating planes and the orthogonal complement of their span. Then J_f is holomorphic.

The theorem follows easily from the lemma.

Questions

- Does the condition of $(n-1)$-isotropicity place restrictions on the conformal structure of the minimal surface, especially for large genus?
- Does the space of $(n-1)$-isotropic minimal surfaces have a nice description?

Theorem (- , E. Nedita)

Let $f\colon \Sigma \to \mathbb{R}^{2n}/\Lambda$ be a stable minimal immersion which is $(n-1)$ isotropic. Then f is holomorphic with respect to . . . .

Similarly, if $f\colon \Sigma \to \mathbb{R}^{2n}$ is a stable minimal immersion which is $(n-1)$ isotropic and if Σ is parabolic in the conformal structure induced by f, then f is holomorphic with respect to . . . .

Proposition

Let $f\colon \Sigma \to \mathbb{R}^{2n}$ be an immersion whose normal bundle ν admits a parallel complex structure J_ν. Let J_Σ be anticlockwise rotation by 90° on $T\Sigma$ and define

$$J := J_\Sigma \oplus J_\nu.$$

Suppose that the second fundamental form maps $T_\Sigma^{1,0}$ to $\nu_{\mathbb{C}}^{1,0}$. Then J is constant and f is holomorphic with respect to J.

Of course, the converse holds.