Methods of Integrable Systems in Geometry

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Commuting Partial Differential Operators

Emma Previato

(Boston University) ep@bu.edu

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Contents:

§**0.** History

We work over the complex numbers

We consider the ring of differential operators:

$$\mathcal{D} = \{\sum_{j=0}^{n} u_j(x)\partial^j,\ u_j \text{ analytic near } x = 0\}$$

$\mathcal{D} \subset \mathcal{P} = \{\sum_{-\infty}^{n} u_j(x)\partial^j\}$, formal pseudodifferential operators. $\mathcal{P}$ is a ring:

$$\partial \circ u = u\partial + u'$$

$$\partial^{-1} \circ u = u\partial^{-1} - u'\partial^{-2} + u''\partial^{-3} - \dots$$

Note: Instead, we could take the coefficients to be formal power series

In concrete examples we may allow meromorphic coefficients, such as the Weierstrass $\wp$ function; or consider instead a translation of the x variable that shifts the origin to a point x_0 where all the coefficients are regular (this is always possible for the "rank-1" centralizers of interest – a fact which I already find nontrivial)

Normalize $L \in \mathcal{D}$ ($\mathcal{D}$ has 2 automorphisms, change of variable and conjugation by a function)

$$L = \partial^n + u_{n-2}(x)\partial^{n-2} + \dots + u_0(x).$$

Burchnall-Chaundy problem [BC]: which L's have centralizer $\mathcal{C}_{\mathcal{D}}(L)$ which is larger than a polynomial ring $\mathbb{C}[L_1]$, $L_1 \in \mathcal{D}$?

• If ord $L > 0$ and $A, B \in \mathcal{D}$ both commute with L, then $[A, B] = 0$; in particular, $\mathcal{C}_{\mathcal{D}}(L)$ is commutative, hence every maximal-commutative subalgebra of $\mathcal{D}$ is a centralizer

In $\mathcal{P}$ any (normalized) L has a unique nth root, $n =$ord$L$, of form $\mathcal{L} = \partial + u_{-1}(x)\partial^{-1} + u_{-2}(x)\partial^{-2} + \dots$

• (I. Schur [Sc]) $\mathcal{C}_{\mathcal{D}}(L) = \{\sum_{-\infty}^{N} c_j \mathcal{L}^j, c_j \in \mathbb{C}\} \cap \mathcal{D}$

• It follows that centralizers are 'curves': their transcendence degree over the field of coefficients is 1, and Spec $\mathcal{C}(L)$ can be regarded as an affine curve X

Examples:

genus-1 case: $\mathbb{C}[\wp(x), \wp'(x)] \cong \mathbb{C}[L, B]$

$$L = \partial^2 - 2\wp(x)$$

$$B = 2(\partial^3 + 3\wp(x)\partial - \frac{3}{2}\wp'(x))$$

$$\wp(x) = \frac{1}{x^2} + \sum_{(n,m)\neq(0,0)} \left(\frac{1}{(x + n\omega_1 + m\omega_2)^2} - \frac{1}{(n\omega_1 + m\omega_2)^2} \right)$$

$$B^2 = 4L^3 - g_2 L - g_3$$

(Ince) the Lamé operator $L = -\partial^2 + a(a+1)\wp(x - x_0)$ (with real, smooth potential), is a solution to the Burchnall-Chaundy problem iff $a \in \mathbb{Z}$ (if a is positive the genus is a).

(Halphen) $\partial^3 + (1-n^2)\wp(x)\partial + \frac{1-n^2}{2}\wp'(x)$ gives another solution, where n is the genus of the curve and the isospectral flow for a parameter y gives a solution to the Boussinesq equation, $u(x,y) = 2\frac{(1-n^2)}{3}\wp(x-cy)$,

$$u_{yy} = u_{xxx} + 6uu_x$$

• **Definition.** The rank of a subset of $\mathcal{D}$ is the greatest common divisor of the orders of all the elements of $\mathcal{D}$.

Roughly stated, Burchnall and Chaundy's characterization of rank-1 commutative subalgebras: they correspond to affine curves, and the 'isospectral' ones corresponding to the same curve correspond to points of $\mathrm{Jac}X \backslash \Theta$. The KP deformations are linear flows on Jacobians, and they can be solved exactly in terms of theta functions.

Aside: Isospectral time deformations

Introduce parameters $\underline{t} = (t_1 = x, t_2, t_3, \ldots)$, the KP hierarchy:

$$\partial_{t_j}\mathcal{L} = [(\mathcal{L}^j)_+, \mathcal{L}]$$

where $(\ )_+$ is projection $\mathcal{P} \to \mathcal{D}$, is a set of PDE's on $u_i(\underline{t})$, which turn out to be commuting Hamiltonian flows (AKS=Adler-Kostant-Symes). A solution $\mathcal{L}$ is "stationary" w.r.t. t_j iff $\mathcal{L}^j \in \mathcal{D}(\Rightarrow \partial_{t_j}\mathcal{L} = 0)$ e.g. for $j = 2$ we get KdV and for $j = 3$ we get Boussinesq, both reductions of the KP equation: $u_{yy} = (u_t + 6uu_x + u_{xxx})_x$ $(y = t_2, t = t_3)$.

More generally, let $K_j = (\mathcal{L}^j)_+$ and say that a KP solution is stationary if a nontrival combination $\sum_1^N c_j\mathcal{L}^j \in \mathcal{D}$, i.e. the corresponding time operator $\sum_1^N c_j K_j$ acts trivially.

An ODO $L \in \mathcal{D}$ is such that the corresponding KP solution is stationary for some N not a multiple of order(L) iff L is a solution to the Burchnall-Chaundy problem (however, this does not guarantee the KP-orbit is finite dimensional unless $\mathcal{C}_{\mathcal{D}}(L)$ has order 1).

Inverse spectral problem (Krichever [K])

We make the following choices:

a Riemann surface Γ of genus g

a point $\infty \in \Gamma$

a local parameter z^{-1} near ∞

a generic divisor $P_1 + \ldots + P_g = D$ (the condition is that $h^0(P_1 + \ldots + P_g - \infty) = 0$, no functions with a zero at ∞ and poles bounded by $P_1 + \ldots + P_g$).

Fact (Krichever). There exists a unique function $\psi(\underline{t}, P)$, the "Baker-Akhiezer (BA) function," satisfying the following conditions:

(i) near ∞, $\psi \sim \exp(\sum_{i \geq 1} t_i z^i)(1 + \sum \xi_i(\underline{t}) z^{-i})$

(ii) at finite points P of the curve, ψ has poles bounded by D and is analytic elsewhere. For such a ψ there exist unique operators K_j such that $K_j \psi = \partial_{t_j} \psi$ and these operators are a solution to the KP hierarchy, in particular $\mathcal{L}\psi = z\psi$ gives $\mathcal{L} \in \mathcal{P}$ as above.

(all statements are local in $\underline{t}$).

$$\psi(\underline{t}) = e^{(\sum_{i \geq 1} t_i (\int_{P_0}^{P} \eta_i - c_i))}.$$

$$\frac{\vartheta(A(P) + \sum_{i \geq 1} U_i t_i + \delta) \vartheta(A(\infty) + \delta - A(D))}{\vartheta(A(P) + \delta - A(D)) \vartheta(A(\infty) + \sum_{i \geq 1} U_i t_i + \delta)}$$

δ is Riemann's constant (to make $\vartheta(A(P) + \delta - A(D))$ vanish for $P = P_j, j = 1, \ldots, g$), $U_i \in \mathbb{C}^g$ suitable vectors (to make ψ into a function of P independent of the path of integration), η_i are suitable meromorphic differentials, $c_i \in \mathbb{C}$ suitable constants (to normalize ψ as in (i) above).

$$\mathcal{L} = \partial + u_{-1}\partial^{-1} + u_{-2}\partial^{-2} + \dots$$

$$u(\underline{t}) = 2\partial_x^2 \log \vartheta(\sum_{j\geq 1} t_j U_j + A(P) + \delta) + \text{const.}$$

with $u = 2u_{-1}$, solves the KP equation.

Remark. The $U_i = \sum_j u_{ij} \frac{\partial}{\partial z_j}$ are linear flows on $\mathrm{Jac}(X)$, so we have linearized the flows of the KP hierarchy. Geometrically, U_1 is the tangent vector to the curve $A(X)$ at $A(\infty)$, and U_j are the j^{th} hyperosculating vectors.

The KP hierarchy is a sequence of deformations for $\mathcal{L} = S\partial S^{-1}$, $S \in \mathcal{P}$, the Baker (eigen)function $\psi = Se^{xz}$ turns differentiation into multiplication.

Question. Classify the commutative subrings of $\mathcal{D}$. The case of rank>1 is open.

There are no BC solutions of order 2 with polynomial coefficients. Dixmier [D] constructs a genus-1 maximal commutative subring of the Weyl algebra in two generators $\mathbb{C}[p,q]$ with multiplication rule defined by the commutator $[p,q]=1$, which can be viewed as a subring of $\mathcal{D}$, by letting $p=\partial$ and $q=x$. Define

$$u = p^3 + q^2 + \alpha,\ v = \frac{1}{2}p,$$

$$L = u^2 + 4v,\ B = u^3 + 3(uv + vu);$$

then $\mathcal{C}(L) = \mathbb{C}[L,B]$ and $B^2 - L^3 = -\alpha$, as shown in [D]. By the assignment $p=\partial,\ q=x$ we obtain $L, B \in \mathcal{D}$ of order 6,9, but the automorphism $\partial \mapsto -x,\ x \mapsto \partial$ will turn the orders into 4,6. Moreover, it will still be true that $\mathcal{C}_{\mathcal{D}}(L) = \mathbb{C}[L,B]$, the affine ring of the curve $\mu^2 = \lambda^3 - \alpha$; in particular, L is a BC solution. These are rings of rank 3, 2, resp. Mironov [M2] computes equations for the coefficients of two rank-2 commuting operators with spectral curve of genus 2, and an example of polynomial-coefficients operators (maximality?)

Question. In rank 1, the motion of poles of the rational KP solutions obeys the Calogero-Moser-Krichever system (ACI Hamiltonian). In higher rank, Veselov in his thesis cf. [V] set up an analogous integrable system. ACI features?

§**1.** Introduction of the problem

To pose the question algebraically again, refer to [BEG] for a concept of "quantum integrable system":

In classical mechanics, an integrable Hamiltonian system on a manifold M of dimension n is a collection of functions $I_1, \ldots, I_n$ on the cotangent bundle $\mathcal{T}^*M$ that pairwise Poisson commute and are functionally independent.

On an n-dimensional algebraic variety M, we regard a dominant map $f : \mathcal{T}^*M \to \Lambda$, Λ an n-dimensional affine variety, whose fibres are Lagrangian, as the analog of $(I_1, \ldots, I_n){:}\mathcal{T}^*M \to \mathbb{A}^n$, and say that the pair (Λ, f) is an integrable Hamiltonian system. This then defines an embedding of rings of regular functions $f^* : \mathcal{O}(\Lambda) \to \mathcal{O}(\mathcal{T}^*M)$, which is a homomorphism of algebras with a Poisson-commuting image.

The quantum analog of the Poisson algebra $\mathcal{O}(\mathcal{T}^*M)$ is the algebra $\mathcal{D}(M)$ of differential operators on M.

Definition [BEG] (i) A quantum completely integrable system (QCIS) on M is a pair (Λ, θ) where Λ is an n-dimensional affine variety and $\theta : \mathcal{O}(\Lambda) \to \mathcal{D}(M)$ is an embedding of algebras. If $\Lambda = \mathbb{A}^n$, such a mapping is defined by an n-tuple $D_1, \ldots D_n$ of differential operators on M which are algebraically independent and pairwise commute

(ii) The eigenvalue problem for (Λ, θ), a system of differential equations:

$$\theta(g)\psi = g(\lambda)\psi,\ \lambda \in \Lambda,\ g \in \mathcal{O}(\Lambda),$$

is a $\mathcal{D}$-module M_λ over M generated by an element $1_\lambda \in M_\lambda$ with the relations $\theta(g) \cdot 1_\lambda = g(\lambda)1_\lambda$, $\forall g \in \mathcal{O}(\Lambda)$. The rank of (Λ, θ) is defined to be the rank of M_λ at the generic point of M, or the dimension of the space of formal solutions of the system. A QCIS is said to be algebraically integrable if it is dominated by another QCIS of rank 1.

A system (Λ, θ) is dominated by one (Λ', θ') if there is a map of algebras $h : \mathcal{O}(\Lambda) \to \mathcal{O}(\Lambda')$ with $\theta = \theta' \circ h$, in which case $M_\lambda \cong \oplus_i M_{\lambda'_i}$

Example: L gives an algebraically integrable system if and only if it can be obtained by the Krichever map (Inverse spectral problem above)

Theorem [BEG] A QCIS is algebraically integrable if and only if the differential Galois group of the corresponding system is commutative, for generic $\lambda \in \Lambda$.

§**2.** Examples I [CV, BEG]: Differential Galois theory

To generalize the Lamé operator:

$L = \frac{d^2}{dx^2} + u(x), u(x) = -\frac{2g(g+1)}{2}\wp(x)$

$$L = -\Delta + \sum_{\alpha\in\Re_+} g_\alpha \wp(\langle\alpha, x\rangle).$$

where $\Re_+$ is the set of positive roots for a simple complex Lie algebra $\mathfrak{g}$ of rank n, $\langle -,-\rangle$ is some positive scalar product in $\mathbb{R}^n$, invariant under the action of the Weyl group, and $g_\alpha = m_\alpha(m_\alpha+1)\langle\alpha,\alpha\rangle$ for some $m_\alpha \in \mathbb{Z}$

For $\mathfrak{g} = A_{n-1}, B_2$,

$$L = -\Delta + 4\sum_{i<j}\frac{1}{(x_i - x_j)^2}$$

$$L = -\partial_1^2 - \partial_2^2 + \frac{2}{x_1^2} + \frac{2}{x_2^2} + \frac{4}{(x_1 - x_2)^2} + \frac{4}{(x_1 + x_2)^2}$$

Definition [CEO] A generalized Lamé operator is a Schrödinger operator $L = -\Delta + u(x)$, $\Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$, with elliptic potential u of the following form

$$u(x) = \sum_{\alpha\in\mathcal{A}} c_\alpha \wp(\alpha(x)|\tau),$$

$\mathcal{A}$ a finite set of affine-linear functions on $V = \mathbb{C}^n$, such that the resulting potential $u(x)$ has the properties of periodicity and quasi-invariance.

Periodicity and quasi-invariance are crucial generalizations of the one-dimensional case (Calogero-Moser potentials); to simplify matters, we assume the former to mean that u is periodic with lattice $\mathcal{L}+\tau\mathcal{L}$ where $\mathcal{L}$ is the dual of the lattice generated by (loosely speaking) the linear part α_0 of the transformations α contained in $\mathcal{A}$. The property of quasi-invariance is an important analytic condition, corresponding to the equations defining the KdV locus in the one-variable case; $u(x)-u(s_\pi x)$ is required to be divisible by $(\alpha(x)-m-n\tau)^{2m_\alpha+1}$, where: $\{x : \ \alpha(x) = m + n\tau\}$ is one of the hyperplanes π comprising the singular locus of $u(x)$, c_α is assumed to be of the form $m_\alpha(m_\alpha + 1)(\alpha_0, \alpha_0)$ for some positive integer m_α, and s_π is the reflection with respect to π.

Theorem [CEO] Any generalized Lamé operator which is completely and strongly integrable, i.e., admits a commutative family of differential operators $L_1 = L, L_2, \ldots, L_n$ which have meromorphic coefficients, are periodic with respect to the same lattice, so that they are defined on the torus $T = \mathbb{C}^n/(\mathcal{L} + \tau\mathcal{L})$ ($\mathcal{L}$ is an appropriate lattice of rank n), and have algebraically independent homogeneous constant highest symbols $s_1, \ldots, s_n$ for which the system

$$s_1(\xi) = \ldots = s_n(\xi) = 0$$

has the unique solution $\xi = 0$, is algebraically integrable.

Examples

$L = -\Delta + \sum_{\alpha \in \Re_+} m_\alpha(m_\alpha + 1)\langle \alpha, \alpha \rangle \wp(\langle \alpha, x \rangle)$

$L = -\Delta + 2m(m+1) \sum_{i<j} (\wp(x_i - x_j) + \wp(x_i + x_j)) + \sum_{i=1}^{n} \sum_{s=0}^{3} g_s(g_s+1)\wp(x_i + \omega_s)$, with ω_s $(s = 0, \ldots, 3)$ denoting the half periods $0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$.

A non-Coxeter example was found in [VFC] (for any numbers l, m):

$L = -\Delta + \frac{l(l+1)}{x_1^2} + \frac{m(m+1)}{x_2^2} + \frac{4(l+m+1)}{(\sqrt{2m+1}x_1 + \sqrt{2l+1}x_2)^2} + \frac{4(l+m+1)}{(\sqrt{2m+1}x_1 - \sqrt{2l+1}x_2)^2}$.

Question Compare the variety Spec $\mathcal{C}(L)$, which is affine, and the "Hermite-Bloch" variety that parametrizes the Bloch eigenfunctions of L.

§3. Algebraic Geometry: Naïve and non-naïve

Recall our emphasis on differential algebra:

$$\mathcal{D} = \mathbb{C}[[x_1, \ldots, x_n]][\partial_1, \ldots, \partial_n] = \left\{ \sum_{|\alpha|=0}^{N} u_\alpha(x)\partial^\alpha \right\},$$

α a multi – index in $\mathbb{N}^n$

Review of the one-variable case

Lemma [BC]. *If $[L, B] = 0$ then there exists a polynomial in two variables $f(\lambda, \mu) \in \mathbf{C}[\lambda, \mu]$ such that $f(L, B) \equiv 0$, if we assign "weight" $na+mb$ to a monomial $\lambda^a \mu^b$ where $n = ord\, L$, $m = ord\, B$, then the terms of highest weight in f are $\alpha\lambda^m + \beta\mu^n$ for some constants α, β.*

Proof and Construction: The idea is that by commutativity B acts on V_λ, the n-dimensional vector space of solutions $y(x)$ of $Ly = \lambda y$ (L is regular); $f(\lambda, \mu)$ is the characteristic polynomial of this operator; to see that $f(L, B) \equiv 0$ it is enough to remark that $f(\lambda, \mu) = 0$ iff L, B have a "common eigenfunction": $\begin{cases} Ly = \lambda y \\ By = \mu y \end{cases}$ hence $f(L, B)$ would have an infinite-dimensional kernel (eigenfunctions belonging to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ are independent by a Vandermonde argument). The algebraic curve is encoded by the "BC matrix" M: if

$$L-\lambda = u_{0,0} + u_{0,1}\partial + \ldots + \partial^n$$

$$(0 = u_{0,n+1} = u_{0,n+2} = \ldots)$$

$$\partial \circ (L-\lambda) = u_{1,0} + u_{1,1}\partial + \ldots$$

$$\ldots$$

$$\partial^{m-1} \circ (L-\lambda) = u_{m-1,0} + \ldots$$

$$B-\mu = u_{m,0} + u_{m,1}\partial + \ldots + \partial^m$$

$$\ldots$$

$$\partial^{n-1} \circ (B-\mu) = u_{m+n-1,0} + \ldots$$

then $M = [m_{ij}]$ with $m_{ij} = u_{i-1,j-1}$ $(i = 1, \ldots, m+n;\ j = 1, \ldots, m+n)$ is such that $\det M = f(\lambda,\mu)^r$, where $r = \gcd(\mathrm{ord}L, \mathrm{ord}B)$.

Example:

$$L = \partial^2 - \frac{2}{x^2}$$

$$B = \partial^3 - \frac{3}{x^2}\partial + \frac{3}{x^3}$$

$$\det \begin{bmatrix} -\lambda - \frac{2}{x^2} & 0 & 1 & 0 & 0 \\ \frac{4}{x^3} & -\lambda - \frac{2}{x^2} & 0 & 1 & 0 \\ -\frac{12}{x^4} & \frac{8}{x^3} & -\lambda - \frac{2}{x^2} & 0 & 1 \\ -\mu + \frac{3}{x^3} & -\frac{3}{x^2} & 0 & 1 & 0 \\ -\frac{9}{x^4} & -\mu + \frac{9}{x^3} & -\frac{3}{x^2} & 0 & 1 \end{bmatrix} = \mu^2 - \lambda^3.$$

Question 1 Does a maximal-commutative subalgebra $\mathcal{A}$ of $\mathcal{D}$ in $n > 1$ variables have to be finitely generated, as in $n = 1$? The answer is no. Notice that when it is, Spec$\mathcal{A}$ can be viewed as an affine variety, and it turns out to have dimension $\leq n$ [BEG].

Question 2 When we have a finitely-generated subalgebra $\mathcal{A}$ of $\mathcal{D}$, is there a way to identify explicitly the variety Spec$\mathcal{A}$, for example by algebraic equations satisfied by the generators, given by a differential resultant, as was the case for dimension 1?

Commutative rings of PDOs

$$\mathcal{D} = \mathbb{C}(x_1, \ldots, x_n)[\partial_1, \ldots, \partial_n]$$

$$\mathcal{D}_0 = \mathbb{C}[\partial_1, \ldots, \partial_n] \subset \mathcal{D}$$

Construction (Kasman-P. [KP]):

A maximal-commutative ring $R \subset \mathcal{D}$ which is not finitely generated.

Step I Suppose $p(\partial_1, \ldots, \partial_n) \in \mathcal{D}_0$ factors $= L \circ K$, $\quad L, K \in \mathcal{D}$

let $R(K) := (K \circ \mathcal{D}_0 \circ K^{-1}) \cap \mathcal{D}$

Theorem $R(K)$ is maximal commutative.

(use $e^{x_1 z_1 + \ldots + x_n z_n}$)

Step II. Let $R_\lambda =$ polynomials q in $\mathbb{C}[x,y]$ s.t. q_x, q_y, q_{xy} are divisible by $(xy-\lambda)$,

$(= \mathbb{C}[\omega_i(xy-\lambda)^3],\ \omega_i$ a basis of $\mathbb{C}[x,y])$

Theorem $R(K) \simeq R_\lambda$

Example:

$$p = (\partial_1\partial_2 - \lambda)^3, \quad K = x_1x_2(\partial_1\partial_2 - \lambda) \circ \frac{1}{x_1x_2}$$

$$L = \partial_1^2\partial_2^2 + \frac{1}{x_1}\partial_1\partial_2^2 - x_1^{-2}\partial_2^2 + \frac{1}{x_2}\partial_1^2\partial_2$$

$$+\frac{1-2\lambda x_1x_2}{x_1x_2}\partial_1\partial_2 + \frac{-1-\lambda x_1x_2}{x_1^2x_2}\partial_2$$

$$-x_2^{-2}\partial_1^2 + \frac{-1-\lambda x_1x_2}{x_1x_2^2}\partial_1 + \lambda^2 + \frac{1}{x_1^2x_2^2} + \frac{\lambda}{x_1x_2}$$

Construction: differential resultant
(F.S. Macaulay for polynomials)

$L_1, \ldots, L_{n+1}$ orders $\ell_1, \ldots, \ell_{n+1}$

$$N = 1 + \sum_i (\ell_i - 1)$$

R: each row $\partial^\alpha \circ (L_i - \mu_i)$ $1 \leq |\alpha| \leq \binom{n+N-l_i}{n}$

Resultant=gcd of all maximal minors = 0 if $L_1, \ldots, L_{n+1}$ commute.
Conjecture: indep. of $x_1, \ldots, x_n$.

The natural question is then, does there exist a commutative ring of scalar PDOs, isomorphic to the functions on an abelian surface, regular off a theta divisor? Maybe the answer are Kleinian functions [BEELS], but I do not know if those Schrödinger operators commute. A natural generalization of the 1-variable theory begins with 2-variable Schrödinger operators $-\Delta + q(x_1, x_2)$, $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$, whose spectrum is in some sense algebro-geometric, with eigenfunctions $\psi(x_1, x_2, z_1, z_2)$ defined on an algebraic surface where z_1, z_2 are local parameters:$(-\Delta + q)\psi = \lambda(z_1, z_2)\psi$, for example with potential essentially a sum of 1-variable algebro-geometric potentials ("separable").

§4. Examples II [S, N, M]: Abelian Surfaces

Sato-Nakayashiki's approach:

$$\partial_i = \frac{\partial}{\partial x_i} \quad 0 \leq i \leq r, \quad |\alpha| = \alpha_0 + \ldots + \alpha_{r-1}$$

$$\mathcal{D} = \left\{ \sum_{|\alpha|<<\infty} a_\alpha(x)\partial^\alpha,\ a_\alpha(x) \in \mathbb{C}[[x]],\ \alpha \in \mathbb{N}^r \right\}$$

Sketch:

micro-differential operators: codirection x_0

$$\mathcal{P} := \left\{ \sum_{|\alpha|<<\infty} a_\alpha(x)\partial^\alpha, \quad \alpha \in \mathbb{Z} \times \mathbb{N}^{r-1} \right\}$$

A a g dimensional Abelian variety, $\Theta \subset A$ a smooth principal polarization

$\nabla_j = \frac{\partial}{\partial x_j} - \int \eta_j \qquad \eta_j$ 2nd kind

$\mathcal{L}$ line bundle over A,

$$\phi_{\mathcal{L}} : \mathcal{A}_\Theta \to M(g! \times g!, \mathcal{D})$$

deformations of $\mathcal{D}$-modules

Nakayashiki [N] produced rings of $(g! \times g!)$ matrices whose entries are differential operators in g variables, isomorphic to the ring $H^0(A \backslash \Theta, \mathcal{O}_A)$ where A is a g-dimensional principally polarized abelian variety, and Θ a smooth theta divisor (note: A cannot be a Jacobian as soon as $g \geq 4$). Nakayashiki defines a sequence of time derivatives in the data, so as to let them flow (linearly on $\mathrm{Pic}A \cong A$) and derive the analog of the KP equations for the ϑ function.

Definition. Let X be a smooth projective variety, $g = \dim H^1(X, \mathcal{O}_X)$, $D \subset X$ an ample divisor, $A^\vee = \overline{H^0(X, \Omega^1_X)}/H^1(X, \mathbb{Z}) = \mathrm{Pic}^0 X$ the Picard variety of X and $\mathcal{P} \to X \times A^\vee$ the Poincaré line bundle. By choosing a standard basis $\{a_1, \ldots, a_g; b_1, \ldots, b_g\}$ of the torsion-free part of $H_1(X, \mathbb{Z})$ and normalized holomorphic differentials $\omega_1, \ldots, \omega_g$ on X, the ring of differential operators $\mathcal{D}_{A^\vee}$ can be identified with $\mathcal{O}_X[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_g}]$, where $(x_1, \ldots, x_g)$ are coordinates on the universal cover of $A^\vee$.

Let $\mathcal{F}(D)(n)$ be the Fourier-Mukai transform of the sheaf $\mathcal{O}_X(nD)$,

$$\mathcal{F}(D)(n) = \pi_{2*}\left(\pi_1^* \mathcal{O}_X(nD) \otimes_{\mathcal{O}_X \otimes \mathcal{O}_{A^\vee}} \mathcal{P}\right)$$

where π_i are the two projections from $X \times A^\vee$, and let $\eta_1, \ldots, \eta_g$ be a normalized basis of differentials of the second kind with poles only on D.

Then $\mathcal{F}(D) = \bigcup_{n=0}^{\infty} \mathcal{F}(D)(n)$ can be given the structure of a $\mathcal{D}_{A^\vee}$-module by the connections ∇_j viewed as operators $\frac{\partial}{\partial x_j} - \int_o^z \eta_j$ on the sections (o, a base point, and z, are points of the universal cover of X). $\mathcal{F}(D)$ with this structure is called a Baker-Akhiezer module. If $\mathcal{F}(D)_{\bar{c}}$ is the stalk at $\bar{c}$, where $\bar{c} \in A^\vee$ is the projection of a point c of the universal cover of $A^\vee$, then the elements of $M_c = \bigcup_n M_c(n)$,

$$M_c(n) = \mathcal{F}(D)_{\bar{c}} \exp\left(-\sum_{i=1}^{g} x_i \int_o^z \eta_i \right),$$

are called Baker-Akhiezer functions. When X is a curve of genus g, these coincide with the Baker-Akhiezer functions defined by Krichever, if D is a point $P_\infty \in X$.

When $X = A$ is a principally polarized abelian variety of dimension g, and Θ a theta divisor, then $A^\vee$ can be identified with A by $a \mapsto [T_a^* \Theta - \Theta]$, where T_a is the translation map. Therefore if $\mathcal{L}_c$ is the bundle over A defined by the cocycle $\rho(m + \Omega n, z) = \exp(2\pi i {}^t m \cdot c)$, where Ω is the period matrix and $c \in \mathbb{C}^g$, and $L_c(n) = H^0(A, \mathcal{L}_c(n\Theta)), L_c = \cup_n L_c(n)$, then the theta functions with characteristics:

$$\vartheta \begin{bmatrix} s/n \\ 0 \end{bmatrix} (nz + c; n\Omega)/\vartheta^n(z), s \in \mathbb{Z}^g/n\mathbb{Z}^g,$$

give a basis of $L_c(n)$, and

$$M_c = \sum_{n=-\infty}^{\infty} \sum_{s \in \mathbb{Z}^g/n\mathbb{Z}^g} \mathcal{O}_A$$

$$\cdot \frac{\vartheta \begin{bmatrix} s/n \\ 0 \end{bmatrix} (nz + c + x; n\Omega)}{\vartheta^n(z)} \cdot \exp\left(- \sum_{i=1}^{g} x_i \zeta_i(z) \right),$$

where $\zeta_i = \frac{\partial}{\partial z_i} \log \vartheta$, and the normalized line bundle $\mathcal{P}$ satisfies $\mathcal{P}_{A \times \{c\}} \cong \mathcal{L}_c$.

If Θ is smooth, then $\mathrm{gr}_{\geq 1} M_c$ is generated over $\mathrm{gr}\mathcal{D}_{A^\vee}$ by $\oplus_{n=1}^{g+1} \mathrm{gr}_n M_c$ for any $c \in \mathbb{C}^g$.

Moreover, if $\bar{c} \neq 0$, $\mathrm{gr}_{\geq 1} M_c$ is a free $\mathrm{gr}\mathcal{D}_{A^\vee}$-module of rank $g!$, and the filtration is such that the action of first-order differential operators satisfies $\mathcal{D}_{A^\vee}^{(1)} \mathcal{F}(\Theta)(i) = \mathcal{F}(\Theta)(i+1)$ for a sufficiently large i.

Then the map

$$i_c = H^0(A \backslash \Theta, \mathcal{O}_A) \to M(g! \times g!, \mathcal{D}_{A^\vee})$$

defined by $f \Psi_c = i_c(f) \Psi_c$, where the vector Ψ_c gives a basis of M_c, gives an isomorphism of the ring of functions on A with poles at most on Θ (of any multiplicity) with a (commutative) ring of $(g! \times g!)$ matrices of differential operators on A. Notice that if we identify $\mathcal{O}_A$ with the ring of convergent power series $\mathbb{C}\{\{x_1, \ldots, x_g\}\}$, the entries of the matrices are elements of the ring of differential operators.

Analog of the KP flows. Sato remarked that there is no natural multivariable generalization of the KP hierarchy, because for the analogous rings $\mathcal{D} = \mathbb{C}[[t_1, \ldots, t_g]][\partial_1, \ldots, \partial_g], \Psi = \mathbb{C}[[t]][[\partial_1^{-1}, \partial_1^{-1}\partial_2 \ldots, \partial_1^{-1}\partial_g]][\partial_1]$ filtered by order $\alpha_1 + \ldots + \alpha_g$ of $\partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_g^{\alpha_g}$, there is no natural choice of a free left $\mathbb{C}[[t]]$ submodule $\mathcal{E}_0$ of $\Psi = \mathcal{J} \oplus \mathcal{E}_0$, for deforming $\mathcal{D}$-submodules $\mathcal{J}$ of Ψ. He then advocated the above described algebro-geometric example, where the "codirection" dt_1 is naturally given by an equation $z_1 = 0$ for the theta divisor, under the identification $\partial_i^{-1} \leftrightarrow z_i$ with local parameters near a point of the spectral variety. Define $\mathcal{P}_{\mathrm{const}}$ under $\partial^\alpha \leftrightarrow \partial^\alpha / (\mathcal{P}t_1 + \ldots + \mathcal{P}t_g)$. Indeed, the following time deformations are defined by Nakayashiki:

$N_{ct}(n) = \sum_{s \in \mathbb{Z}^g / n\mathbb{Z}^g} \mathbb{C}[[t]] \cdot$

$$\frac{\vartheta \begin{bmatrix} s/n \\ 0 \end{bmatrix} (nz + c - (x' \cdot d - x_1, x'))}{\vartheta^n(z)} \exp$$

$$\left(-\sum_{i=0}^{g} \sum_{n \geq \delta_{i1}} t_{n,(i)} \frac{(-1)^n}{n!} \left(u_{n,(i)} + d_i(1 - \delta_{i1}) u_{n+1,(1)} \right) \right),$$

where we set $x_1 = t_{1,(1)}, x_i = t_{1,(i)}, 2 \leq i \leq g, d = (d_2, \ldots, d_g) \in \mathbb{C}^{g-1}$, and denote by x' the vector $(x_2, \ldots, x_g)$ if $x = (x_1, \ldots, x_g)$; $u_{i_1, \ldots, i_g}$ denotes $\partial_{z_1}^{i_1} \ldots \partial_{z_g}^{i_g} \log \vartheta(z)$.

N_{ct} can be embedded in $\mathcal{P}$ as a $\mathcal{D}$-submodule, $\varphi \in N_{ct} \mapsto \iota(\varphi) = W_\varphi$, in such a way that $W_{\partial\varphi/\partial x_i} = \frac{\partial W_\varphi}{\partial x_i} + W_\varphi \partial_i = \partial_i W_\varphi$ for $1 \leq i \leq g$ and the $\mathcal{D}$-submodule of $\mathcal{P}$, $\mathcal{J}_{ct}(n) = \iota\left(B_{ct}(n+1)\right)$, satisfies $\Psi^{(n)} = \mathcal{J}_{ct}(n) \oplus \Psi(J_{n,ct})$ where $J_{n,ct}$ is a suitable collection of indices from $\mathbb{Z} \times \mathbb{N}^{g-1}$, and
$\Psi(J) = \{\sum a_\alpha \partial^\alpha | a_\alpha = 0 \text{ unless } \alpha \in J\}$.

Then: $\frac{\partial W_\alpha}{\partial t_\beta} + W_\alpha \partial^\beta \in \mathcal{J}_{ct} = \cup_{n=0}^{\infty} \mathcal{J}_{ct}(n)$, for β in the complementary index set, and are suitable $\mathbb{C}[[t]]$-generators of $\mathcal{J}_{ct}$ of the form $\partial^\alpha +$[an operator whose terms have multiindices belonging to $J_{|\alpha|,ct}$].

Remark. As observed by Mironov [M1], the functions in $N_{ct} \cdot \exp$

$$\left(-\sum_{i=0}^{g} \sum_{n \geq \delta_{i1}} t_{n,(i)} \frac{(-1)^n}{n!} \left(u_{n,(i)} + d_i(1-\delta_{i1}) u_{n+1,(1)} \right) \right)$$

are independent of the time variables, so there isn't really a deformation beyond the g-dimensional variety $A^\vee$, which indeed is $\mathrm{Pic}^0 A$.

Barsotti Equations [B]: ϑ is a theta function if and only if $P_{2r}(\vartheta)$ span a finite-dimensional vector space,

$$\vartheta(u+v)\vartheta(u-v) = 2\vartheta^2(u) \sum_{r=0}^{\infty} P_{2r}(\vartheta(u)) v^{2r}$$

§5. Examples III: Sato's τ function

A.N. Parshin proposed a different construction, based on the theory of higher local fields, in which the commuting partial differential operators are scalar.

Parshin's ring contains Sato's ring properly; the two constructions have not yet been compared, but this should be possible since Parshin gave a geometric interpretation for the rings [P1-2].

An n-dimensional local field K (with "last" residue field $\mathbb{C}$) is the field of iterated Laurent series $K = \mathbb{C}((x_1))\dots((x_n))$, with structure of a complete discrete valuation ring $\mathcal{O} = \mathbb{C}((x_1))\dots((x_{n-1}))[[x_n]]$ with residue field an $(n-1)$-dimensional local field. Notice that the order of the variables matters, in the sense that $\mathbb{C}((x_1))((x_2))$ does not contain the same elements of $\mathbb{C}((x_2))((x_1))$, e.g., although they are isomorphic. These are suited to give local coordinates on an n-dimensional manifold, since the inverse of a polynomial in x_1, x_2, say, can be written as the inverse of the highest-order monomial times something entire, so as a Laurent series it is bounded in both variables. Whereas, the symbols

$$\mathbb{C}((x_1, x_2)) = \left\{ \sum_{|i+j|<N} c_{ij} x_1^i x_2^j \right\}$$

cannot be given a ring structure unless we want to define sums of infinitely many complex numbers, because $i + j = N$ involves infinitely many indices unless we bound j (or i) from above. With this definition, Parshin constructs a $2n$-dimensional skew-field $\mathcal{P}$, infinite-dimensional over its center, namely the (formal) pseudo-differential operators:

$$\mathcal{P} = \mathbb{C}((x_1))\ldots((x_n))((\partial_1^{-1}))\ldots((\partial_n^{-1})).$$

The order of the variables is also singled out in the definition of the grading:

If $L = \sum_{i \le m} a_i \partial_n^i$ with $a_m \neq 0$, we say that the operator L has order m and write $\mathrm{ord} L = m$.

If $P_i = \{L \in \mathcal{P} | \mathrm{ord} L \le i\}$, then $\ldots P_{-1} \subset P_0 \subset \ldots$ is a decreasing filtration of $\mathcal{P}$ by subspaces and $\mathcal{P} = P_+ \oplus P_-$, where $P_- = P_{-1}$ and P_+ consists of operators involving only nonnegative powers of ∂_n.

The highest term (h.t.) of an operator L is defined by induction on n: if $L = \sum_{i \le m} a_i \partial_n^i$ and $\mathrm{ord} L = m$, then $h.t.(L) = h.t.(a_m) \cdot \partial^m$. If $h.t.(L) = f \partial_1^{m_1} \ldots \partial_n^{m_n}$ with $0 \neq f \in \mathbb{C}((x_1))\ldots((x_n))$, then we let $\nu(L) = (m_1, \ldots, m_n)$. We consider also the subring $E = \mathbb{C}[[x_1, \ldots, x_n]]((\partial_1^{-1}))...((\partial_g^{-1}))$ of $\mathcal{P}$, and $E_\pm = E \cap \mathcal{P}_\pm$.

E is much larger than Sato's ring

$$\mathbb{C}[[x_1, \ldots, x_n]][[\partial_1^{-1}, \partial_1^{-1}\partial_2, \ldots, \partial_1^{-1}\partial_g]]$$

when $n > 1$.

FACT (Parshin 1999):

• An operator $L \in E$ is invertible in E if and only if the coefficient f in the highest-order term of L is invertible in the ring $\mathbb{C}[[x_1, \ldots, x_n]]$. If the highest-order term f in $L \in \mathcal{P}$ is an m-th power in $\mathbb{C}((x_1))\ldots((x_n))$ (resp., $\mathbb{C}[[x_1, \ldots, x_n]]$ for $L \in E$) then there exists, unique up to multiplication by m-th root of unity, an operator $M \in \mathcal{P}$ (resp. $M \in E$) such that $M^m = L$. Thus, P_0 is a discrete valuation ring in $\mathcal{P}$ with residue field

$$\mathbb{C}((x_1))\ldots((x_n))((\partial_1^{-1}))\ldots((\partial_{n-1}^{-1})).$$

• Let $L_1 \in \partial_1 + E_-, \ldots, L_n \in \partial_n + E_-$. Then $[L_i, L_j] = 0$ for all i, j if and only if there exists an operator $S \in 1 + E_-$ such that $L_i = S^{-1}\partial_i S$, for all i.

• For $L = (L_1, \ldots, L_n)$ as above, the flows

$$\frac{\partial L}{\partial t_m} = ([(L_1^{m_1} \cdots L_n^{m_n})_+, L_1] \ldots [(L_1^{m_1} \cdots L_n^{m_n})_+, L_n])$$

$$m = (m_1, \ldots, m_n) \in \mathbb{Z}_{\geq 0} \times \ldots \times \mathbb{Z}_{\geq 0}$$

commute, and if $S \in 1 + \mathcal{P}_-$ satisfies

$$\frac{\partial S}{\partial t_m} = -\left(S\partial_1^{m_1} \cdots \partial_n^{m_n} S^{-1}\right)_- S,$$

then $L = \left(S\partial_1 S^{-1}, \ldots, S\partial_n S^{-1}\right)$ evolves according to them.

Fourier transforms as maps to (formal) Grassmannians

Nakayashiki's construction of differential operators allows us to generalize the Krichever map in a way that Parshin uses to extend it from curves to surfaces.

Parshin views the Krichever map as a consequence of quasi-isomorphism of complexes:

$$\Gamma(X\backslash p, \mathcal{F}) \oplus \hat{\mathcal{F}}_p \to \hat{\mathcal{F}}_p \otimes_{\hat{\mathcal{O}}_p} K_p$$

(X is a curve, $p \in X$ a smooth point, $\mathcal{F}$ a torsion-free rank-r sheaf on X). Then if z is a formal local parameter at p, and e_p a trivialization of $\mathcal{F}$ at p, we have $\hat{\mathcal{O}}_p = \mathbb{C}[[z]]$, $K_p = \mathbb{C}((z))$ and canonical identifications: $\Gamma(X\backslash p, \mathcal{F}) \subset \mathcal{F}_\eta \otimes_{\hat{\mathcal{O}}_p} K_p = \hat{\mathcal{F}}_p \otimes_{\hat{\mathcal{O}}_p} K_p = \hat{\mathcal{O}}_p^{\oplus r} \otimes_{\hat{\mathcal{O}}_p} K_p = \mathbb{C}((z))^{\oplus r}$.

Then $W = \Gamma(X\backslash p, \mathcal{F}) \subset V = \mathbb{C}((z))^{\oplus r}$ is the (Krichever) map to Sato's Grassmannian.

When X is a (projective, irreducible) surface and C a curve on X and p a smooth point on the curve, $x \in C$, let:

$$B_x(\mathcal{F}) = \bigcap_{D \neq C} ((\hat{\mathcal{F}}_x \otimes K_x) \cap (\hat{\mathcal{F}}_x \otimes \hat{\mathcal{O}}_{x,D})), \text{ and}$$

$$B_C(\mathcal{F}) = (\hat{\mathcal{F}}_C \otimes K_C) \cap (\bigcap_{x \neq p} B_x),$$

for example, when $X = \mathbb{P}^2$, $C = \mathbb{P}^1$, then $B_p(\mathcal{O}_X) = \mathbb{C}[[u]]((t))$, $B_C(\mathcal{O}_X) = \mathbb{C}[u^{-1}]((u^{-1}t))$, $\hat{\mathcal{O}}_{p,C} = \mathbb{C}((u))[[t]]$, $\hat{\mathcal{O}}_p = \mathbb{C}[[u,t]]$.

The analog of the Krichever map sends the data:

$$(X, C, p \in C, z_1, z_2, \mathcal{F}, e_p),$$

with local equation for C given by $z_2 = 0$ near p, to a pair of subspaces $B = B_C(\mathcal{O}_X) \subset K, W = B_C(\mathcal{F}) \subset V$ as above, with $K = \mathbb{C}((z_1))((z_2))$, attendant to two complexes whose cohomologies are isomorphic to $H^{\cdot}(X, \mathcal{O}_X)$ and $H^{\cdot}(X, \mathcal{F})$.

Baker function

Lee [L1-3] was able gave a Sato-Grassmannian interpretation of Parshin's equations.

The Baker function associated to an operator $S \in 1 + \mathcal{P}_-$ is defined to be $\psi(t,z) = S\exp(\xi(t,z))$ where $\xi(t,z) = \sum_{\alpha_i \geq 0} t_\alpha z^\alpha$, for an auxiliary set of variables z_α. Since

$$\partial^\alpha \exp(\xi(t,z)) = z^\alpha \exp(\xi(t,z)),$$

the Baker function has the form

$$\psi(t,z) = \tilde{\psi}(t,z)\exp(\xi(t,z))$$

where $\tilde{\psi}(t,z)$ is a formal power series in $z_1, \ldots, z_n$ and $\frac{\partial}{\partial t_m}\psi = (L^m)_+\psi$ for an L that flows as in the KP hierarchy.

For a vector $(s_1, \ldots, s_n) \in \mathbb{C}^n$, the operator $G(s)$ on functions $f(t,z)$ is defined by $G(s)f(t,z) = f((t_\alpha - \alpha^{-1}s^{-\alpha})_\alpha, z)$ with the multiindex notation

$$\alpha^{-1}s^{-\alpha} = \alpha_1^{-1} \ldots \alpha_n^{-1} s_1^{-\alpha_1} \ldots s_n^{-\alpha_n}$$

(here $\alpha_i > 0$). A τ-function for the operator S is a formal solutions of the equation $\psi(t,z) = \tilde{\psi}(t,z)\exp(\xi(t,z)) = \frac{G(z)\tau(z)}{\tau(t)}\exp\xi(t,z)$. It can be shown that such a function exists by solving recursively for its coefficients.

Tau function

Segal and Wilson gave an analytic model of Sato's Grassmannian, consisting of certain subspaces of the Hilbert space of square-integrable functions on the n-torus.

Let $H = L^2(S^1)$ be the Hilbert space consisting of all square-integrable functions on $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. H can be written in the form

$$H = \langle z^\alpha \mid \alpha \in \mathbb{Z} \rangle_{\mathbb{C}},$$

with the customary notation for the Hilbert-space span, and the usual inner product. H can be decomposed into the sum of two orthogonal subspaces, $H = H_+ \oplus H_-$, $H_- = \langle z^{-1}, ..., z^{-m}, ...\rangle$ and $H_+ = \langle 1, z, ..., z^m, ...\rangle$ (here m is any positive integer). The Segal-Wilson Grassmannian $\mathrm{Gr}(H)$ consists of the subspaces W of H such that the projection p_+ of W to H_+ is Fredholm, and the projection p_- to H_- is Hilbert-Schmidt. The formal Grassmannian consists of the subspaces W "commensurable" with H_+, in the sense that $\dim(W + H_+)/W \cap H_+$ is finite, or equivalently [S], $\dim(W \cap H_-) = \dim H/(W + H_-) < \infty$.

An analogous construction can be given for the multivariable case. We fix a positive integer n and denote by $z = (z_1, \ldots, z_n)$ the coordinate function for $\mathbb{C}^n$. The multi-index notation is defined as follows: $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ if $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$. If $\beta = (\beta_1, \ldots, \beta_n)$ is another element of $\mathbb{Z}^n$, we write $\alpha \leq \beta$ when $\alpha_i \leq \beta_i$ for each $i \in \{1, \ldots, n\}$.

Let $H = L^2(T^n)$ be the Hilbert space consisting of all square-integrable functions on the n-torus

$$T^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_1| = \cdots = |z_n| = 1\},$$

which can be identified with the product of n copies of the unit circle $S^1 \subset \mathbb{C}^n$. Then the Hilbert space H can be written in the form

$$H = \langle z^\alpha | \alpha \in \mathbb{Z}^n \rangle_{\mathbb{C}}.$$

We define a splitting $H = H_+ \oplus H_-$ adapted to Parshin's filtration and Krichever map [P1-3].

Then, as in the one-variable case, there is a one-to-one correspondence between certain subspaces of H commensurable to

$$H_+ := \mathbb{C}[[z_1, ..., z_n]]$$

and wave functions, given by $\psi \mapsto W$, where a spanning set for W is given by all derivatives $\partial_1^{j_1} ... \partial_n^{j_n} \psi$ with $j \epsilon \mathbf{0}$, evaluated at $\mathbf{t} = \mathbf{0}$. We take this to be the Grassmannian $\mathrm{Gr}(H)$.

Again, H has a decomposition of the form

$$H = H_+ \oplus H_-$$

where H_- has basis the set of monomials complementary to H_+. We denote by $p_+ : H \to H_+$ and $p_- : H \to H_-$ the natural projection maps. A subspace W of H is said to be transversal to H_- if the restriction $p_+ |_W : W \to H_+$ of p_+ to W is an isomorphism.

We consider a holomorphic function $g : D^n \to \mathbb{C}$ defined on the closed polydisk

$$D^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \| z_1| \leq 1, \ldots, |z_n| \leq 1\}$$

with $g(\mathbf{0}) = \mathbf{1}$, where $\mathbf{0} = (0, \ldots, 0) \in \mathbb{Z}^n$. If we set

$$\mathbb{Z}^n_+ = \{\alpha \in \mathbb{Z}^n | \alpha \geq \mathbf{0},\ \alpha \neq \mathbf{0}\},$$

then $g(z) = g(z_1, \ldots, z_n)$ can be written in the form

$$g(z) = \exp\Bigg(\sum_{\alpha \in \mathbb{Z}^n_+} t_\alpha z^\alpha\Bigg)$$

with $t_\alpha \in \mathbb{C}$ for all $\alpha \in \mathbb{Z}^n_+$.

We define the maps $\mu_g, \mu_{g^{-1}} : H \to H$ by

$$(\mu_g f)(z) = g(z) f(z), \quad (\mu_{g^{-1}} f)(z) = g(z)^{-1} f(z)$$

for all $f \in W$ and $z \in \mathbb{C}^n$. Since $\mu_{g^{-1}}(H_+) \subset H_+$, with respect to the decomposition of H, the map $\mu_{g^{-1}}$ can be represented by a block matrix of the form

$$\mu_{g^{-1}} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix},$$

whose entries are the maps

$$a : H_+ \to H_+, \quad b : H_- \to H_+, \quad c : H_- \to H_-.$$

Let Γ_+ denote the space of holomorphic functions $g : D^n \to \mathbb{C}$ with $g(\mathbf{0}) = 1$. Given $W \in \mathrm{Gr}(H)$, we set

$$\Gamma_+^W = \{g \in \Gamma_+ | \mu_{g^{-1}} W \text{ is transversal to } H_-\}.$$

Thus g belongs to Γ_+^W if and only if the map $p_+|_{\mu_{g^{-1}}W} : \mu_{g^{-1}} W \to H_+$ is an isomorphism.

Let $\mathcal{S}$ be the complex vector space of formal Laurent series in $z_1^{-1}, \ldots, z_n^{-1}$ consisting of series of the form

$$v = \sum_{\alpha \leq \nu} f_\alpha(t) z^\alpha$$

for some $\nu \in \mathbb{Z}^n$ with $t = (t_\alpha)_{\alpha \in \mathbb{Z}_+^n}$. We consider the subspace $\mathcal{S}_-$ of $\mathcal{S}$ consisting of the series which can be written as

$$v = \sum_{k=-\infty}^{k_0} f_k(t; z_1, \ldots, z_{n-1}) z_n^k$$

for some $k_0 \in \mathbb{Z}$ with $k_0 \leq -1$, so that there is a decomposition of the form

$$\mathcal{S} = \mathcal{S}_+ \oplus \mathcal{S}_-,$$

where $\mathcal{S}_+$ consists of the series of the form

$$\sum_{k=0}^{\ell_0} f_k(t; z_1, \ldots, z_{n-1}) z_n^k$$

for some nonzero integer ℓ_0.

Given an element W of the Grassmannian $\mathrm{Gr}(H)$, the associated Baker function $w_W(g, z)$ is the function defined for $g \in \Gamma_+^W$ and $z \in T^n$ satisfying the conditions

$$w_W(g, z) \in W, \quad \mu_{g^{-1}} w_W(g, z) = 1 + u$$

with $u \in \mathcal{S}_-$.

Since each element $g \in \Gamma_+^W$ can be written in the exponential form, the Baker function $w_W(g, z)$ may be regarded as a function for $t = (t_\alpha)_{\alpha \in \mathbb{Z}_+^n}$ and $z \in T^n$. Thus we may write

$$w_W(g, z) = w_W(t, z),$$

$t = (t_\alpha)_{\alpha \in \mathbb{Z}_+^n}$.

Let $W \in \mathrm{Gr}(H)$ be transversal to H_-, so that the map $p_+|_W : W \to H_+$ is an isomorphism, and let g be an element of Γ_+^W. We consider the sequence

$$H_+ \overset{(p_+|_W)^{-1}}{\to} W \overset{\mu_{g^{-1}}}{\to} \mu_{g^{-1}}W \overset{p_+}{\to} H_+ \overset{\mu_g}{\to} H_+$$

of complex linear maps.

Given $g \in \Gamma_+^W$ and an element $W \in \mathrm{Gr}(H)$ transversal to H_-, the associated τ-function $\tau_W(g) = \tau_W(t) = \tau_W((t_\alpha)_{\alpha \in \mathbb{Z}_+^n})$ is the function

$$\tau_W(g) = \det\bigl(\mu_g \circ p_+ \circ \mu_{g^{-1}} \circ (p_+|_W)^{-1}\bigr)$$

given by the determinant of the composite of the linear maps above.

Let $\Lambda : H_+ \to H_-$ be the linear map given by $\Lambda = p_- \circ (p_+|_W)^{-1}$. Then the τ-function can be written in the form

$$\tau_W(g) = \det(1 + a^{-1}b\Lambda),$$

where a and b are as above and 1 denotes the identity map on H_+.

We define the rational numbers ϵ_α for $\alpha \in \mathbb{Z}_+^n$ by requiring

$$\sum_{\alpha \in \mathbb{Z}_+^n} \epsilon_\alpha x^\alpha =$$

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left(\sum_{\beta \in \mathbb{Z}_+^n} x^{\beta} \right)^k ,$$

where $x = (x_1, \ldots, x_n)$ is a multivariable.

Theorem [LP] Let $W \in \mathrm{Gr}(H)$ be transversal, and let $g : D^n \to \mathbb{C}$ be an exponential.

Then the associated τ-function

$$\tau_W(g) = \tau_W((t_\alpha)_{\alpha \in \mathbb{Z}_+^n})$$

satisfies

$$\mu_{g^{-1}} w_W(g, z) = \frac{\tau_W\big((t_\alpha + \epsilon_\alpha z^{-\alpha})_{\alpha \in \mathbb{Z}_+^n}\big)}{\tau_W((t_\alpha)_{\alpha \in \mathbb{Z}_+^n})},$$

where $w_W(g, z)$ is the Baker function.

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