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$$\mathcal{D}\psi = \left[\begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}\right]\psi = 0$$

where $\partial = \frac{\partial}{\partial z}$ and z is a conformal parameter

Given a torus $\Sigma \approx \mathbb{R}^2/\Lambda$, the Floquet function:

$$\mathcal{D}\varphi = 0, \quad \varphi(q+\gamma) = \mu(\gamma)\varphi(q),$$

where $q \in \mathbb{R}^2$, $\gamma \in \Lambda$.

The spectral curve:

$$\Gamma = \{P(\mu_1, \mu_2) = 0\} \subset \mathbb{C}^2$$

where $\mu_i = \mu(\gamma_i)$ and γ_1,γ_2 generate Λ.

From Γ and z we reconstruct

$$E(\Sigma) = \int_M UV \frac{i dz \wedge d\bar{z}}{2} =$$

for $\mathbb{R}^3$ $= \frac{1}{4}\mathcal{W} = \frac{1}{4}\int_M (H^2 + \widehat{K}) d\mu.$

$$f : \Sigma \to G$$

$$d_{\mathcal{A}} : \Omega^1(\Sigma; E) \to \Omega^2(\Sigma; E),$$

where $E = f^{-1}(TG) \xrightarrow{\pi} \Sigma$ is the pulllback of TG and

$$\begin{aligned}
\omega &= u dz + u^* d\bar{z}, \\
d_{\mathcal{A}}\omega &= d'_{\mathcal{A}}\omega + d''_{\mathcal{A}}\omega, \\
d'_{\mathcal{A}}\omega &= -\nabla_{\bar{\partial} f} u dz \wedge d\bar{z}, \\
d''_{\mathcal{A}}\omega &= \nabla_{\partial f} u^* dz \wedge d\bar{z}.
\end{aligned}$$

The derivational equations:

$$d_{\mathcal{A}}(df) = 0,$$

$$d_{\mathcal{A}}(*df) = i e^{2\alpha} H N dz \wedge d\bar{z}$$

where N is the normal field, H is the mean curvature, and $e^{2\alpha} dz d\bar{z}$ is the metric.

$$\Psi = f^{-1}\partial f, \quad \Psi^* = f^{-1}\bar{\partial} f$$

$$\partial\Psi^* - \bar{\partial}\Psi + \nabla_\Psi \Psi^* - \nabla_{\Psi^*}\Psi = 0,$$

$$\partial\Psi^* + \bar{\partial}\Psi + \nabla_\Psi \Psi^* + \nabla_{\Psi^*}\Psi = e^{2\alpha} H f^{-1}(N)$$

For $SU(2)$ and $H = 0$ we have

$\bar{\partial}\Psi - \partial\Psi^* + [\Psi^*, \Psi] = 0, \quad \bar{\partial}\Psi + \partial\Psi^* = 0,$

i.e. the following connection on $F^{-1}(TG)$:

$$\mathcal{A}_\lambda = \left(\partial + \frac{1+\lambda^{-1}}{2}\Psi, \bar{\partial} + \frac{1+\lambda}{2}\Psi^*\right),$$

is flat for all λ.

Thus we have the "integrability."

$$\langle \Psi, \Psi \rangle = \langle \Psi^*, \Psi^* \rangle = 0, \quad \langle \Psi, \Psi^* \rangle = \frac{1}{2} e^{2\alpha}$$

$$\Psi = \sum_{k=1}^{3} Z_k e_k, \quad \Psi^* = \sum_{k=1}^{3} \bar{Z}_k e_k$$

$$Z_1^2 + Z_2^2 + Z_3^2 = 0$$

$$Z_1 = \frac{i}{2}(\bar{\psi}_2^2 + \psi_1^2), \quad Z_2 = \frac{1}{2}(\bar{\psi}_2^2 - \psi_1^2),$$

$$Z_3 = \psi_1 \bar{\psi}_2$$

$$f_z = f\Psi$$

$$ds^2 = e^{2\alpha} dz d\bar{z}, \quad e^{\alpha} = (|\psi_1|^2 + |\psi_2|^2),$$

$$\omega = A dz^2, \quad A = \langle \nabla_{f_z} f_z, N \rangle.$$

We consider the following Lie groups with Thurston's geometries:

$$\mathbb{R}^3, \quad SU(2), \quad \text{Nil}\,, \quad \widetilde{SL}_2, \quad \text{Sol}\,.$$

The bases e_1, e_2, e_3 in the Lie algebras are as follows:

1) Nil and $\widetilde{SL}_2$ admits rotations around the "e_3-axis";

2) $[e_1, e_2] = 0$ for Sol.

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$$\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} \longmapsto \begin{pmatrix}-\bar{\psi}_2 \\ \bar{\psi}_1\end{pmatrix}$$

$$U = V = \frac{1}{2} H e^{\alpha} \qquad \text{for } \mathbb{R}^3$$

$$U = \bar{V} = \frac{1}{2}(H - i) e^{\alpha} \qquad \text{for } SU(2)$$

$$U = V = \frac{1}{2} H e^{\alpha} + \frac{i}{4}(|\psi_2|^2 - |\psi_1|^2) \qquad \text{for Nil}$$

$$U = \frac{1}{2} H e^{\alpha} + i(\frac{1}{2}|\psi_1|^2 - \frac{3}{4}|\psi_2|^2),$$
$$V = \frac{1}{2} H e^{\alpha} + i(\frac{3}{4}|\psi_1|^2 - \frac{1}{2}|\psi_2|^2) \qquad \text{for } \widetilde{SL}_2$$

$$U_{\text{Sol}} = \frac{1}{2} H e^{\alpha} + \frac{1}{2}\bar{\psi}_2^2 \frac{\bar{\psi}_1}{\psi_1},$$
$$V_{\text{Sol}} = \frac{1}{2} H e^{\alpha} + \frac{1}{2}\bar{\psi}_1^2 \frac{\bar{\psi}_2}{\psi_2} \qquad \text{for Sol}$$

Minimal surfaces:

$$\bar\partial\psi_1 = \partial\psi_2 = 0 \qquad \text{for } \mathbb{R}^3$$

$$\bar\partial\psi_1 = -\frac{i}{2}(|\psi_1|^2 + |\psi_2|^2)\psi_2,$$
$$\partial\psi_2 = \frac{i}{2}(|\psi_1|^2 + |\psi_2|^2)\psi_1 \qquad \text{for } SU(2)$$

$$\bar\partial\psi_1 = \frac{i}{4}(|\psi_2|^2 - |\psi_1|^2)\psi_1,$$
$$\partial\psi_2 = -\frac{i}{4}(|\psi_2|^2 - |\psi_1|^2)\psi_1 \quad \text{for Nil}$$

$$\bar\partial\psi_1 = i\left(\frac{3}{4}|\psi_1|^2 - \frac{1}{2}|\psi_2|^2\right)\psi_2,$$
$$\partial\psi_2 = -i\left(\frac{1}{2}|\psi_1|^2 - \frac{3}{4}|\psi_2|^2\right)\psi_1 \quad \text{for } \widetilde{SL}_2$$

$$\bar\partial\psi_1 = \frac{1}{2}\bar\psi_1^2\bar\psi_2, \quad \partial\psi_2 = -\frac{1}{2}\bar\psi_1\bar\psi_2^2 \quad \text{for Sol}$$

Constant mean curvature surfaces:

$$A_{\bar{z}} = 0 \qquad \text{for } \mathbb{R}^3, SU(2)$$

$$\left(A + \frac{{Z_3}^2}{2H + i} \right)_{\bar{z}} = 0 \qquad \text{for Nil}$$

$$\left(A + \frac{5}{2(H - i)} Z_3^2 \right)_{\bar{z}} = 0 \qquad \text{for } \widetilde{SL}_2$$

? for Sol

- a surface is CMC in $\mathbb{R}^3, SU(2)$, and Nil iff the equation holds
- if a surface in $\widetilde{SL}_2$ is CMC then the equation holds (similar equations are known for $S^2 \times \mathbb{R}$ and $H^2 \times \mathbb{R}$ (Abresch–Rosenberg))
- the equations for Nil and $\widetilde{SL}_2$ were derived by Abresch

The energy:

$$\frac{1}{4}\int H^2 d\mu \qquad \text{for } \mathbb{R}^3$$

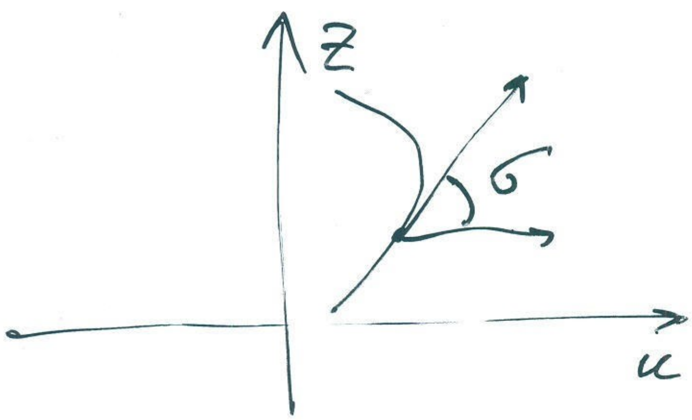
$$\frac{1}{4}\int (H^2+1) d\mu \qquad \text{for } SU(2)$$

$$\frac{1}{4}\int \left(H^2+\frac{\widehat{K}}{4}-\frac{1}{16}\right) d\mu \qquad \text{for Nil}$$

$$\frac{1}{4}\int \left(H^2+\frac{5}{16}\widehat{K}-\frac{1}{4}\right) d\mu \qquad \text{for } \widetilde{SL}_2$$

? for Sol

$$V_H = \frac{4\pi}{H} - \frac{4H^2-3}{4H} S_H \quad , H >> 0$$



$$ds^2 = du^2 + \frac{4u^2}{4u^2+u^4} dv^2$$

$$E = \int UV \, d\mu = \frac{\pi}{8} \int \left(\dot{\sigma} - \frac{\sin\sigma}{u}\right)^2 \sqrt{4u^2+u^2} \, ds$$

+ topological term
$\left(= \frac{1}{4} \chi(M)\right)$

CMC revolution sphere $\Leftrightarrow$ $\dot{\sigma} = \frac{\sin\sigma}{u}$

$$\Delta H + 2H(H^2-K) + 2e^{-4\alpha}(A\bar{z}_3^2 + \bar{A}z_3^2) = 0$$