Critical behavior and the limit distribution for long-range oriented percolation (OP)

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(Joint work with L.-C. Chen)

Contents

1. Long-range OP with characteristic exponent α

OP on $\mathbb{Z}^d \times \mathbb{Z}_+$ (= discrete-time model for the spread of an infection)

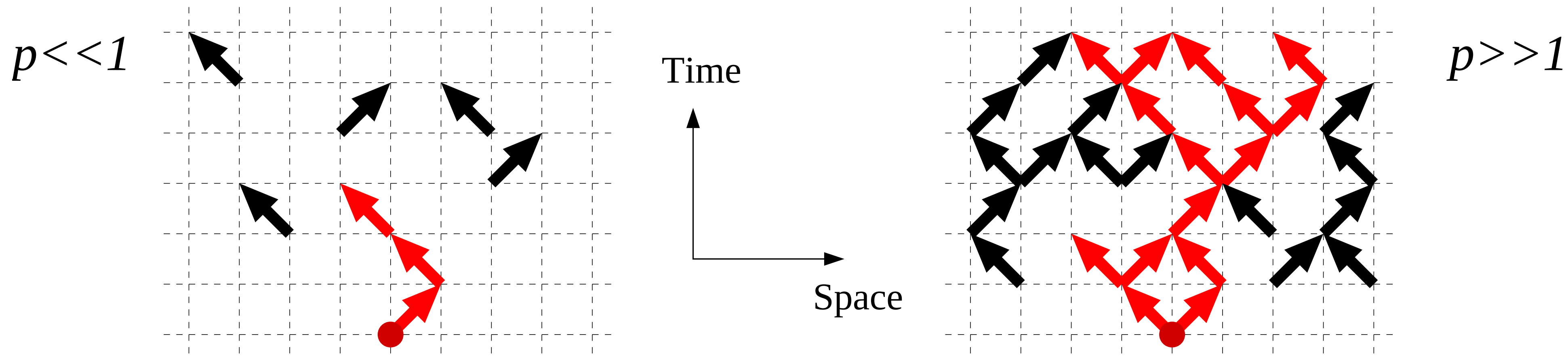
- $((x,n),(y,n+1))$: (time-oriented) bond $\begin{cases} \text{occupied} & \text{w.p. } pD(x-y) \\ \text{vacant} & \text{w.p. } 1-pD(x-y) \end{cases}$

  (D : $\mathbb{Z}^d$-symm. probab. distr. on $\mathbb{Z}^d$; $p \in [0, \|D\|_\infty^{-1}]$: percolation parameter)

- $\{(x,n) \rightarrow (y,l)\} \stackrel{\text{def}}{=} \left\{ \begin{array}{c} (x,n)=(y,l), \text{ or } \exists\{(w_j,j)\}_{j=n}^{l} \text{ with } w_n = x \text{ and} \\ w_l = y \text{ s.t. } ((w_j,j),(w_{j+1},j+1)) \text{ occupied } \forall j \end{array} \right\}$

  ($\rightarrow$: connected to)

e.g., 1-dimensional nearest-neighbour model: $D(x) = \frac{1}{2}\mathbb{1}\{|x|=1\}$



Existence of a phase transition (Aizenman-Newman 1984)

$$\exists p_c \geq 1 \text{ s.t. } \sum_{n=0}^{\infty} \sum_{x \in \mathbb{Z}^d} \mathbb{P}_p\Big((o,0) \to (x,n)\Big) \begin{cases} < \infty & \text{if } p < p_c \\ = \infty & \text{if } p \geq p_c \end{cases}$$

(i) Finite-variance OP $\Big(1 - \hat{D}(k) \asymp |k|^2\Big)$

(a) Nearest-neighbour model: $D(x) = \frac{1}{2d}\,\mathbb{1}\{|x|=1\}$

(b) Spread-out model: Kac potential defined by

$$D(x) = \frac{h(x/L)}{\sum_{y \in \mathbb{Z}^d} h(y/L)} = O(L^{-d})\, h(x/L) \quad \text{for some } L < \infty$$

h : bounded probab distribution on $\mathbb{R}^d$ with finite $2+\epsilon^{\text{th}}$ moment

(ii) Long-range OP with exponent α $\Big(1 - \hat{D}(k) \asymp |k|^{\alpha \wedge 2} \,(\times \log \frac{1}{|k|} \text{ if } \alpha = 2)\Big)$

Kac potential defined by h (b.p.d. on $\mathbb{R}^d$) with $h(x) \asymp |x|^{-d-\alpha}$ for $|x| \gg 1$

$$\text{e.g.,} \quad D(x) = \frac{(|\frac{x}{L}| \vee 1)^{-d-\alpha}}{\sum_{y \in \mathbb{Z}^d} (|\frac{y}{L}| \vee 1)^{-d-\alpha}} \quad \text{for some } L < \infty$$

2. Main results

Observables: 2-point function and its Fourier transform

$$\varphi_p(x,n) \stackrel{\text{def}}{=} \mathbb{P}_p\Big((o,0) \to (x,n)\Big) \qquad\qquad Z_p(k;n) \stackrel{\text{def}}{=} \sum_{x\in\mathbb{Z}^d} e^{ik\cdot x}\varphi_p(x,n)$$

Fact $\exists m_p = \lim_{n\uparrow\infty} Z_p(0;n)^{-1/n} \begin{cases} \text{non-increasing in } p>0 \\ >1 \quad \text{for } p\in(0,p_\text{c}) \quad (\because \text{Simon-Lieb ineq}) \end{cases}$

($\because$ sub-additivity)

Theorem (with L.-C. Chen, 2007) Let $d>2(\alpha\wedge 2)$, $\epsilon \in \Big(0, 1\wedge \frac{d-2(\alpha\wedge 2)}{\alpha\wedge 2}\Big)$ and $\lambda \stackrel{\text{def}}{=} L^{-d} \ll 1$. Then, the following (a)–(c) hold for $p\le p_\text{c}$:

(a) $m_{p_\text{c}} = 1, \quad p_\text{c} = 1 + \underbrace{\frac{1}{2}\sum_{n=2}^{\infty} D^{*2n}(o)}_{O(\lambda)} + O(\lambda^2)$

(b) $\exists C = 1+O(\lambda)$ s.t. $Z_p(0;n) = C m_p^{-n}\Big(1+O(n^{-\epsilon})\Big)$

$\Big($cf., $Z_p(0;n) = p^n$ and $p_\text{c} = 1$ for the branching process.$\Big)$

(c) Suppose that $\exists v \in (0,\infty)$ s.t.

$$1-\hat{D}(k) \underset{|k|\to 0}{\sim} \begin{cases} v|k|^{\alpha\wedge 2} & \text{if } \alpha \neq 2 \\ v|k|^2 \log\frac{1}{|k|} & \text{if } \alpha = 2 \end{cases}$$

Let $k_n = \frac{k}{(vn)^{\frac{1}{\alpha\wedge 2}}} \times \begin{cases} 1 & \text{if } \alpha \neq 2, \\ \log\sqrt{n} & \text{if } \alpha = 2. \end{cases}$ Then, $\exists C, C' = 1 + O(\lambda)$ s.t.

$$e^{-C|k|^{\alpha\wedge 2}} \leq \liminf_{n\to\infty} \frac{Z_p(k_n;n)}{Z_p(0;n)} \leq \limsup_{n\to\infty} \frac{Z_p(k_n;n)}{Z_p(0;n)} \leq e^{-C'|k|^{\alpha\wedge 2}}$$

Remark

Finite-variance OP for $d > 4$ (N-Y 1995 (finite-range); vdH-Sl 2002 ($p = p_c$))

$$\text{e.g.,} \quad \exists C = 1 + O(\lambda) \text{ s.t. } \lim_{n\to\infty} \frac{Z_p(k/\sqrt{vn};n)}{Z_p(0;n)} = e^{-C|k|^2}$$

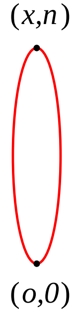
cf., RG/numerical results for $d \leq 4$ (e.g., Hyunggyu-Lucian-SuChan 2005):

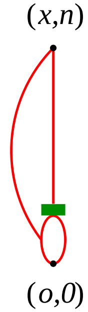
$$Z_{p_c}(0;n) \approx \begin{cases} n^{\eta(d)} & \text{if } d < 4 \\ \log^{1/6} n & \text{if } d = 4 \end{cases}$$

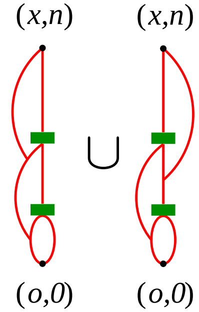
3. Lace expansion

Expansion and its coefficients

$$\varphi_p(x,n) = \pi_p(x,n) + \sum_{l=1}^{n} \sum_{u,v\in\mathbb{Z}^d} \pi_p(u,l-1)\,pD(v-u)\,\varphi_p(x-v,n-l)$$

$$\pi_p(x,n) = \sum_{N=0}^{\infty} (-1)^N \pi_p^{(N)}(x,n) = \mathbb{P}_p\Bigg($$


$$\Bigg) - \mathbb{P}_p\Bigg($$


$$\Bigg) + \mathbb{P}_p\Bigg($$


$$\Bigg) - \cdots$$

cf., Random-walk 2-point function

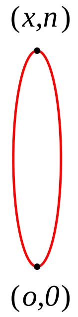
$$P_p(x,n) \overset{\text{def}}{=} (pD)^{*n}(x) = \delta_{x,o}\delta_{n,0} + \sum_{v\in\mathbb{Z}^d} pD(v)\,P_p(x-v,n-1)$$

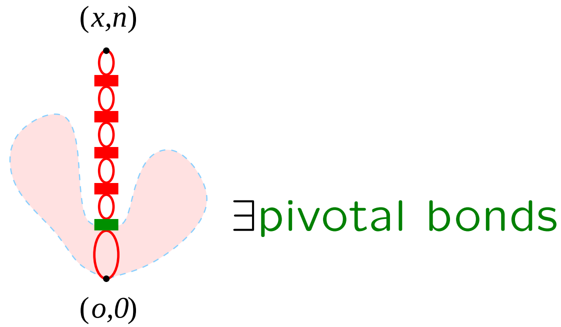
$$\Rightarrow \quad \hat{P}_p(k,z) \overset{\text{def}}{=} \sum_{n=0}^{\infty} \sum_{x\in\mathbb{Z}^d} z^n e^{ik\cdot x} P_p(x,n) = \underbrace{\sum_{n=0}^{\infty} \sum_{x\in\mathbb{Z}^d} (pz)^n e^{ik\cdot x} P_1(x,n)}_{\hat{P}_1(k,pz)}$$

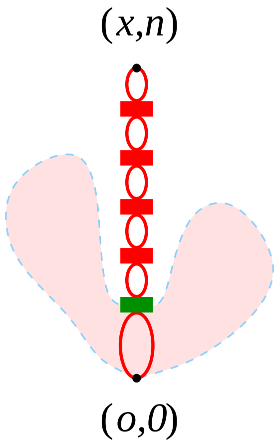
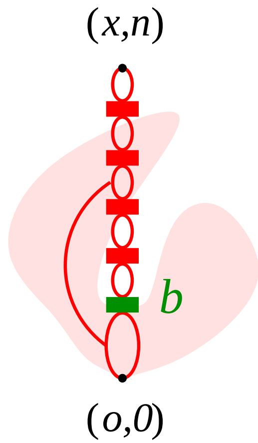
$$\overset{\because pm_p=1}{=} \big(pm_p - pz\hat{D}(k)\big)^{-1} = \big(p(m_p - z) + pz(1-\hat{D}(k))\big)^{-1}$$

First stage $\quad\left(\varphi_p = \pi_p^{(0)} + \pi_p^{(0)} * pD * \varphi_p - R_p^{(1)}\right)$

- $\Big\{(o,0) \to (x,n)\Big\} = \underbrace{\Big\{(o,0) \rightrightarrows (x,n)\Big\}} \;\dot\cup\; \underbrace{\Big\{(o,0) \to (x,n),\ \ (o,0) \not\rightrightarrows (x,n)\Big\}}$

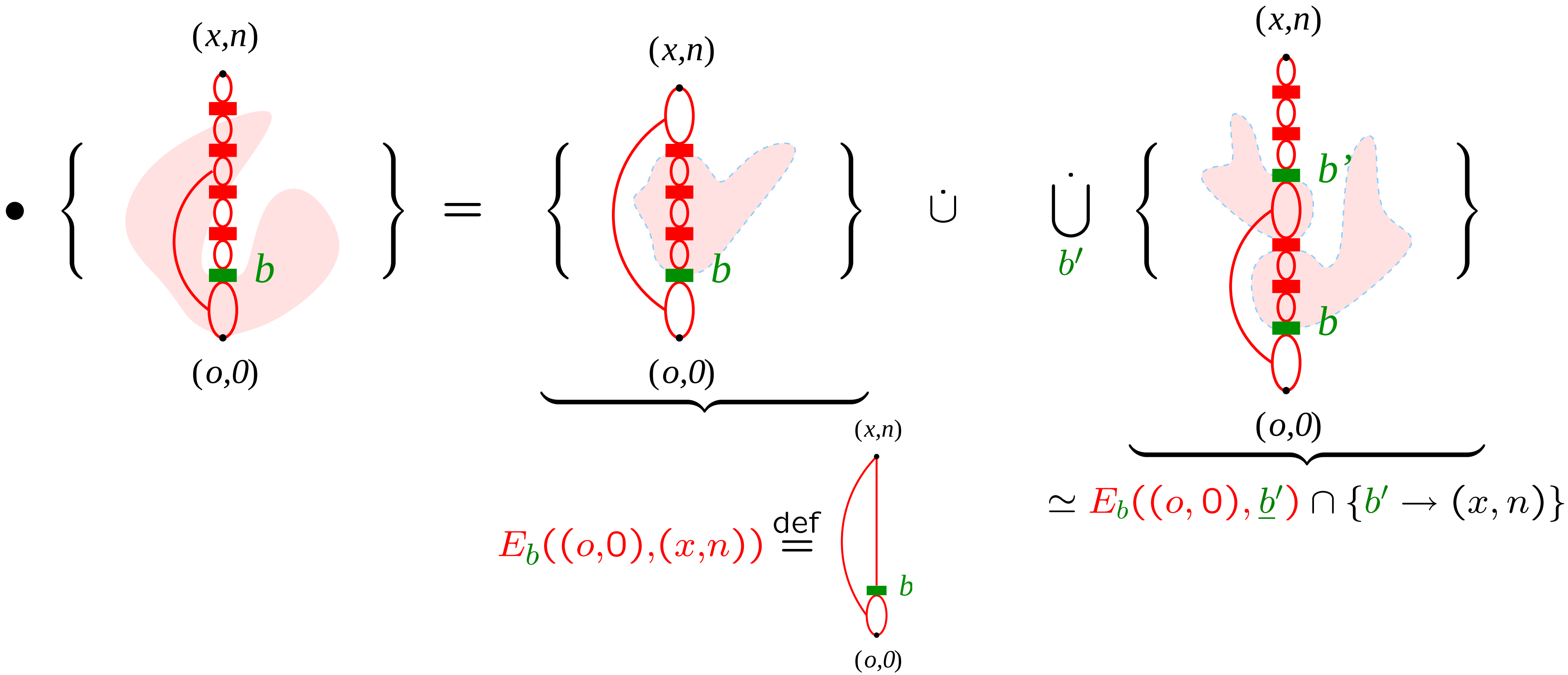




- $\Bigg\{$  $\Bigg\} = \dot{\bigcup_b} \Bigg\{ \Big\{(o,0) \rightrightarrows b \to (x,n)\Big\} \setminus \Bigg\{$  $\Bigg\}\Bigg\}$

- $\mathbb{P}_p\Big((o,0) \rightrightarrows b \to (x,n)\Big) \overset{\text{Markov}}{=} \underbrace{\mathbb{P}_p\Big((o,0) \rightrightarrows \underline{b}\Big)}_{\pi_p^{(0)}} \underbrace{\mathbb{P}_p\Big(b \to (x,n)\Big)}_{pD \cdot \varphi_p}$

Second stage $\left(R_p^{(1)} = \pi_p^{(1)} + \pi_p^{(1)} * pD * \varphi_p - R_p^{(2)}\right)$



- $\mathbb{P}_p\Big(E_b((o,0),\underline{b'}) \cap \{b' \to (x,n)\}\Big) \stackrel{\text{Markov}}{=} \underbrace{\mathbb{P}_p\Big(E_b((o,0),\underline{b'})\Big)}_{\pi_p^{(1)}} \underbrace{\mathbb{P}_p\Big(b' \to (x,n)\Big)}_{pD \cdot \varphi_p}$

4. Analysis of the lace expansion

(i) $0<\widehat{\pi}_p(k,z)<\infty$ uniformly in $k\in[-\pi,\pi]^d$, $|z|\le m_p$

$$\Rightarrow\quad \widehat{\varphi}_p(k,z)=\sum_{n=0}^{\infty}z^n Z_p(k;n)=\Big(\tfrac{1}{\widehat{\pi}_p(k,z)}-pz\widehat{D}(k)\Big)^{-1}$$

$$=\Big(\tfrac{1}{\widehat{\pi}_p(k,z)}\underbrace{-\tfrac{1}{\widehat{\pi}_p(0,m_p)}+pm_p}_{=0}-pz\widehat{D}(k)\Big)^{-1}$$

$$=\Big(p(m_p-z)+pz(1-\widehat{D}(k))+\tfrac{\widehat{\pi}_p(0,m_p)-\widehat{\pi}_p(k,z)}{\widehat{\pi}_p(k,z)\,\widehat{\pi}_p(0,m_p)}\Big)^{-1}$$

(ii) $\left\{\begin{array}{l}\widehat{\pi}_p(0,m_p)-\widehat{\pi}_p(0,z)=O(\lambda)\,p(m_p-z)\\ \widehat{\pi}_p(0,z)-\widehat{\pi}_p(k,z)=O(\lambda)\,pz(1-\widehat{D}(k))\end{array}\right\}$ uniformly in k and $|z|\le m_p$

$$\overset{\lambda\ll 1}{\Rightarrow}\quad \widehat{\varphi}_p(k,z)=\Big(\big(1+O(\lambda)\big)\,p(m_p-z)+\big(1+O(\lambda)\big)\,pz(1-\widehat{D}(k))\Big)^{-1}$$

(iii) $\widehat{\varphi}_p(k,z)\simeq\widehat{P}_1(k,\mu_{p,z})$ for some $\mu_{p,z}$ $\underset{\text{BK ineq}}{\overset{d>2(\alpha\wedge 2)}{\overset{\lambda\ll 1}{\Rightarrow}}}$ Assumptions in (i)–(ii)

(iv) Tauberian estimates: $\widehat{\varphi}_p(0,z)\sim A\big(1-\frac{z}{m_p}\big)^{-1}\ \Rightarrow\ Z_p(0;n)\sim A m_p^{-n}$

Key estimates (with L.-C. Chen, 2007)

- $$\hat\pi_p(0,m_p)-\hat\pi_p(0,z)=(m_p-z)\Big(\underbrace{\partial_z\hat\pi_p(0,z)\Big|_{z=m_p}+O((m_p-z)^\epsilon)}_{\text{Need }\sum_{n=0}^\infty\sum_{x\in\mathbb{Z}^d} n^{1(+\epsilon)}m_p^n\,\pi_p(x,n)}\Big)$$

- For finite-variance OP

$$\hat\pi_p(0,z)-\hat\pi_p(k,z)=\frac{|k|^2}{2d}\Big(\underbrace{-\nabla^2\hat\pi_p(k,z)\Big|_{k=0}+O(|k|^\epsilon)}_{\text{Need }\sum_{n=0}^\infty\sum_{x\in\mathbb{Z}^d}|x|^{2(+\epsilon)}m_p^n\,\pi_p(x,n)}\Big)$$

- For long-range OP with characteristic exponent α

$$\hat\pi_p(0,z)-\hat\pi_p(k,z)\overset{?}{=}\frac{|k|^\alpha}{2d}\Big(-\nabla^\alpha\hat\pi_p(k,z)\Big|_{k=0}+O(|k|^\epsilon)\Big)$$

Adaptation of the trigonometric techniques (B-Cha-vdH-Sl-Sp 2005)

$$\Big|\hat\pi_p(0,z)-\hat\pi_p(k,z)\Big|\le\Big(1-\hat D(k)\Big)\underbrace{\sum_{n=0}^\infty\sum_{x\in\mathbb{Z}^d}m_p^n\Big(\pi_p\text{ with ``extra vertex''}\Big)(x,n)}_{=O(\lambda)\text{ if } d>2(\alpha\wedge2)\text{ and }\lambda\ll1}$$

5. Concluding remark

- Critical behaviour for long-range OP with characteristic exponent α

$$h(x) \asymp |x|^{-d-\alpha} \quad \overset{\substack{d>2(\alpha\wedge 2)\\ \lambda\equiv L^{-d}\ll 1}}{\Rightarrow} \quad \begin{cases} p_c \underset{L\uparrow\infty}{\simeq} 1 + \frac{1}{2}\sum_{n=2}^{\infty} D^{*2n}(o) \\ Z_p(0;n) \underset{n\uparrow\infty}{\sim} (1+O(\lambda))\, m_p^{-n} \\ \frac{Z_p(k_n;n)}{Z_p(0;n)} \underset{n\uparrow\infty}{\simeq} e^{(1+O(\lambda))|k|^{\alpha\wedge 2}} \end{cases}$$

- Key points
  - Lace expansion for OP
  - Application of trigonometric techniques
  - Fractional moments for the time variable of the expansion coefficients

- Open problem

  Fractional moments for the spatial variable of the expansion coefficients