Random walks and the lace expansion II

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Abstract

We give an introduction to the lace expansion with emphasis on recent applications to:
1) the enumeration of self-avoiding walks, and
2) the analysis of random walks on the incipient infinite cluster for oriented percolation.

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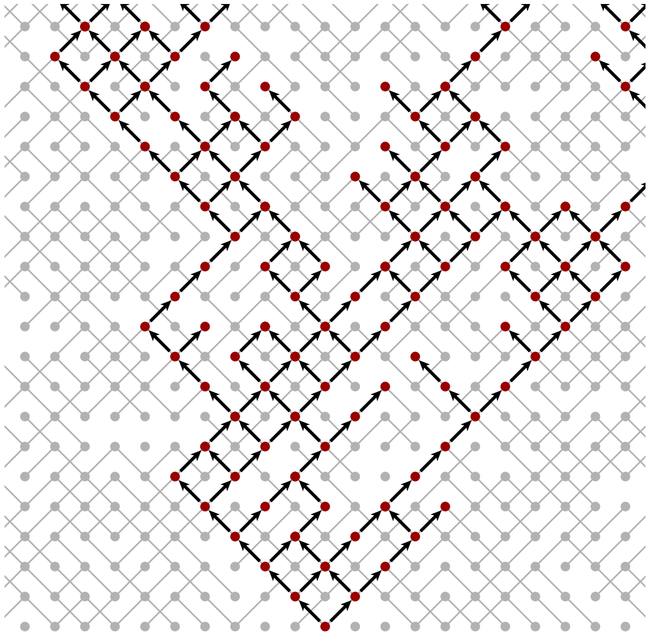
Spread-out oriented percolation

Bond percolation on the directed graph with vertices (x, n), $x \in \mathbb{Z}^d$, $n = 0, 1, 2, \ldots$ and directed bonds $((x, n), (y, n+1))$ with $\|x - y\| \leq L$ (later: L large).

Bonds are independently "occupied" with probability p and otherwise "vacant."



$p = 0.7 > p_c \approx 0.645$

Oriented percolation has a helpful Markov property.

Phase transition for oriented percolation

We say that $(0,0) \longrightarrow (x,n)$ if there is an occupied oriented path from $(0,0)$ to (x,n), and let

$$C(0,0) = \{(x,n) : (0,0) \longrightarrow (x,n)\}.$$

Let $\theta(p) = \mathbb{P}_p(|C(0,0)| = \infty)$. Phase transition: $\exists\, p_c = p_c(d,L) \in (0,1)$ s.t.

$$\theta(p) = 0 \quad \text{if } p < p_c,$$

$$\theta(p) > 0 \quad \text{if } p > p_c.$$

For $p = p_c$: Bezuidenhout–Grimmett (1990) proved $\theta(p_c) = 0$.

Critical behaviour for $p \approx p_c$ well understood for $d > 4$ (when $L \gg 1$).

Oriented percolation: r-point functions

Fix $p = p_c$. The r-point functions ($r \geq 2$) are given by:

$$\begin{aligned} \tau^{(r)}_{\vec{n}}(\vec{x}) &= \tau^{(r)}_{n_1,\dots,n_{r-1}}(x_1.\dots,x_{r-1}) \\ &= \mathbb{P}\Big((0,0) \to (x_1,n_1),\dots,(0,0) \to (x_{r-1},n_{r-1})\Big). \end{aligned}$$

Their Fourier transforms are

$$\hat{\tau}^{(r)}_{\vec{n}}(\vec{k}) = \sum_{\vec{x}\in\mathbb{Z}^{d(r-1)}} \tau^{(r)}_{\vec{n}}(\vec{x}) e^{i\vec{k}\cdot\vec{x}}.$$

The two-point function $\tau = \tau^{(2)}$ obeys

$$\hat{\tau}_n(0) = \sum_{x\in\mathbb{Z}^d} \mathbb{E} I[(0,0) \to (x,n)] = \mathbb{E}|C(0,0) \cap (\mathbb{Z}^d \times \{n\})|$$

and, for $p \neq p_c$,

$$\hat{\tau}_n(0;p) \to \begin{cases} 0 & (p < p_c) \\ \infty & (p > p_c). \end{cases}$$

The two-point function: Results

Theorem. van der Hofstad – S 2003 Let $d > 4$, $p = p_c$, $\delta \in (0, 1 \wedge \frac{d-4}{2})$.
$\exists\ L_0(d)$, $A(d,L)$, $D(d,L)$, $C_i(d)$ such that for $L \geq L_0$ we have

$$\hat{\tau}_n\Big(\frac{k}{\sqrt{Dn}}\Big) = Ae^{-k^2/2d}\left[1 + O\Big(\frac{k^2}{n^\delta}\Big) + O\Big(\frac{1}{n^{(d-4)/2}}\Big)\right]$$

$$C_1 L^{-d} n^{-d/2} \leq \sup_x \tau_n(x) \leq C_2 L^{-d} n^{-d/2}.$$

First line was proved previously by Nguyen and Yang (1995), with a weaker error estimate. Our proof is different and also yields the second line.

In particular, $\lim_{n\to\infty} \hat{\tau}_n(0) = A$
(expected number of sites in cluster of origin at time n).

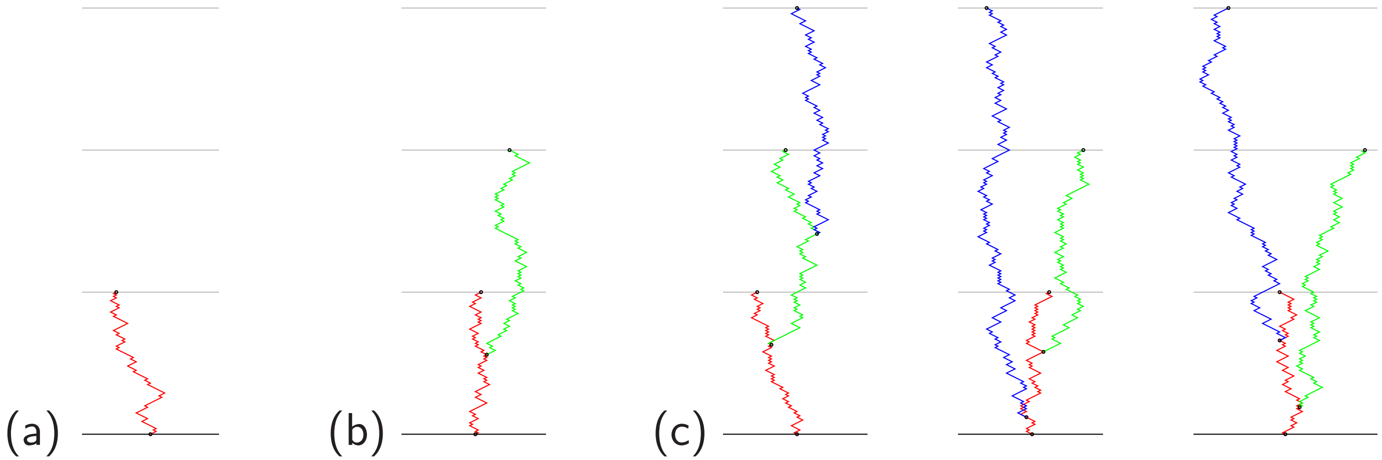
The r-point functions: Results

Let $\hat{p}_t(k) = e^{-k^2 t/2d}$.

Theorem. van der Hofstad – S 2003 Let $d > 4$, $p = p_c$, $\delta \in (0, 1 \wedge \frac{d-4}{2})$, $t_i \in (0,\infty)$. $\exists\ V(d, L)$ such that for $L \geq L_0$ we have

$$\hat{\tau}^{(3)}_{\lfloor n\vec{t}\rfloor}\Big(\frac{\vec{k}}{\sqrt{Dn}}\Big) = nVA^3\left[\int_0^{t_1 \wedge t_2} \hat{p}_s(k_1+k_2)\hat{p}_{t_1-s}(k_1)\hat{p}_{t_2-s}(k_2)\, ds + O\Big(\frac{1}{n^\delta}\Big)\right]$$

with a similar result for all r-point functions, $r \geq 4$ (convergence to super-Brownian motion).



Notation

$$(x, n) = \boldsymbol{x}$$

$$\tau_n(x) = \tau(\boldsymbol{x})$$

$$(\sigma * \tau)(\boldsymbol{x}) = \sum_{\boldsymbol{y}} \sigma(\boldsymbol{y})\tau(\boldsymbol{x} - \boldsymbol{y}) = \sum_{y \in \mathbb{Z}^d} \sum_{m=0}^{n} \sigma_m(y)\tau_{n-m}(x - y)$$

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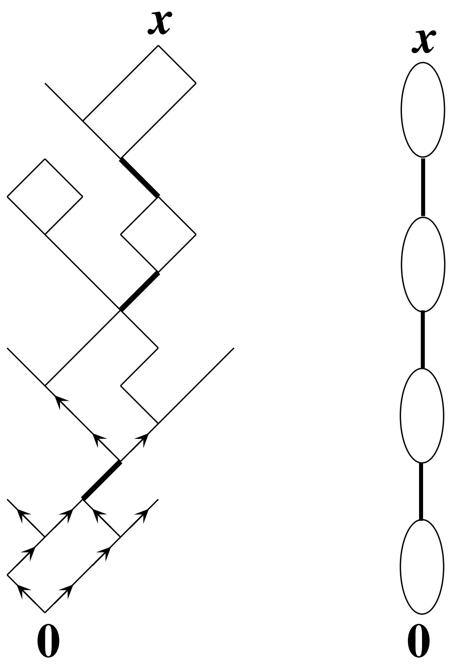
Lace expansion for two-point function



A bond configuration, and its schematic representation as a "string of sausages."

The "string" consists of the pivotal bonds.

The lace expansion is an inclusion-exclusion expansion in which the "sausages" are treated as independent, to leading order.

Lace expansion for two-point function

For oriented percolation, $\exists$ three different versions of the expansion for the two-point function:

(i) Hara–S, (ii) Nguyen–Yang, (iii) Sakai.

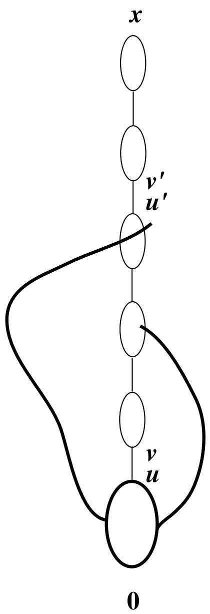
Idea of (i) is the following:

$$
\begin{aligned}
\tau(\boldsymbol{x}) &= \mathbb{P}(\mathbf{0} \Rightarrow \boldsymbol{x}) + \mathbb{P}(\mathbf{0} \rightarrow \boldsymbol{x},\ \mathbf{0} \not\Rightarrow \boldsymbol{x}) \\
&= \mathbb{P}(\mathbf{0} \Rightarrow \boldsymbol{x}) + \sum_{(\boldsymbol{u},\boldsymbol{v})} \mathbb{P}(\mathbf{0} \Rightarrow \boldsymbol{u} \text{ and } (\boldsymbol{u},\boldsymbol{v}) \text{ is occupied and pivotal for } \mathbf{0} \rightarrow \boldsymbol{x}) \\
&= \mathbb{P}(\mathbf{0} \Rightarrow \boldsymbol{x}) + \sum_{(\boldsymbol{u},\boldsymbol{v})} \mathbb{P}(\mathbf{0} \Rightarrow \boldsymbol{u}) \tau_1(\boldsymbol{v}-\boldsymbol{u}) \tau(\boldsymbol{x}-\boldsymbol{v}) - R(\boldsymbol{x}).
\end{aligned}
$$

Write $\mathbb{P}(\mathbf{0} \Rightarrow \boldsymbol{x}) = \delta_{\mathbf{0},\boldsymbol{x}} + \pi^{(0)}(\boldsymbol{x})$. For $\boldsymbol{x} \neq \mathbf{0}$,

$$
\tau(\boldsymbol{x}) = \pi^{(0)}(\boldsymbol{x}) + (\tau_1 * \tau)(\boldsymbol{x}) + (\pi^{(0)} * \tau_1 * \tau)(\boldsymbol{x}) - R(\boldsymbol{x}).
$$

Lace expansion for two-point function



Depiction of the remainder term. The heavy and light lines correspond to percolation clusters living on distinct copies of the same lattice.
Repeated expansion gives

$$\tau(\boldsymbol{x}) = \pi(\boldsymbol{x}) + (\tau_1 * \tau)(\boldsymbol{x}) + (\pi * \tau_1 * \tau)(\boldsymbol{x}),$$

with $\pi(\boldsymbol{x}) = \pi^{(0)}(\boldsymbol{x}) - \pi^{(1)}(\boldsymbol{x}) + \pi^{(2)}(\boldsymbol{x}) - \cdots$.

Induction

Rewrite expansion as:

$$\tau_{n+1}(x) = \sum_y \tau_1(y)\tau_n(x-y) + \pi_{n+1}(x) + \sum_{m=2}^{n}\sum_{u,v} \pi_m(u)\tau_1(v-u)\tau_{n-m}(x-v)$$

and take Fourier transform:

$$\hat{\tau}_{n+1}(k) = \hat{\tau}_1(k)\hat{\tau}_n(k) + \hat{\pi}_{n+1}(k) + \hat{\tau}_1(k)\sum_{m=2}^{n} \hat{\pi}_m(k)\hat{\tau}_{n-m}(k). \qquad (*)$$

Simplest contribution to π_m is $\pi_m^{(0)}$ and, by the BK inequality,

$$|\hat{\pi}_m^{(0)}(k)| = \left|\sum_x \mathbb{P}((0,0) \Rightarrow (x,m))e^{ik\cdot x}\right| \le \sum_x \tau_m(x)^2 \le \|\tau_m\|_\infty \|\tau_m\|_1 \le \|\hat{\tau}_m\|_1 \hat{\tau}_m(0).$$

A general inductive approach to recursions like (*) is given in van der Hofstad – S '02, and leads to the conclusion that $\hat{\tau}_n(k/\sqrt{n})$ behaves like a Gaussian, if $d > 4$ and $L \gg 1$.

Critical branching process vs oriented percolation

Difference: multiple occupancy for branching process
vs
single occupancy for oriented percolation.

Difference not noticeable until two oriented percolation paths join, e.g., $(0,0) \Rightarrow (x,n)$.

For $d > 4$, $L \geq L_0$, $n \geq 1$,

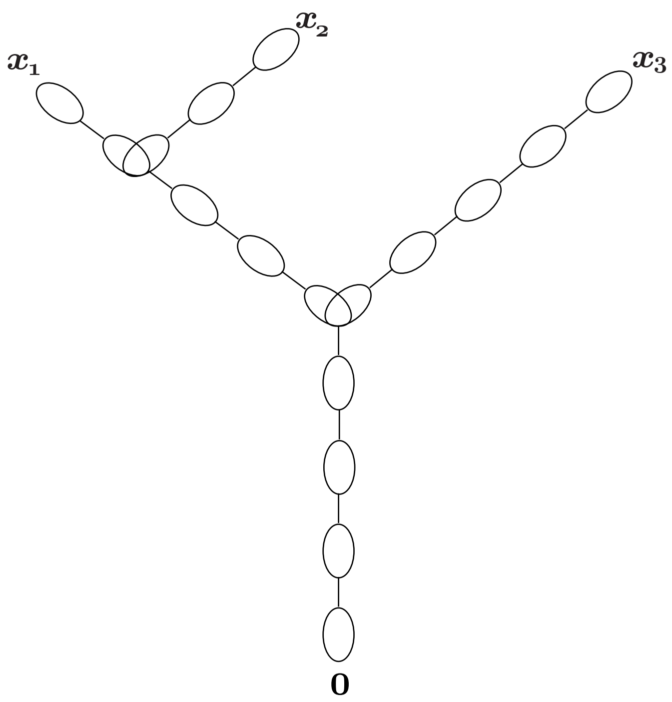
$$\sum_{x \in \mathbb{Z}^d} \mathbb{P}_{p_c}((0,0) \Rightarrow (x,n)) \leq CL^{-d} n^{-d/2},$$

so that

$$\sum_{n=1}^{\infty} n \sum_{x \in \mathbb{Z}^d} \mathbb{P}_{p_c}((0,0) \Rightarrow (x,n)) \leq CL^{-d}.$$

In this sense, the effect of closed loops is small when $d > 4$, and the difference is small.

General r-point functions



A more sophisticated expansion shows that the general r-point function, which can be represented as a “tree of sausages,” asymptotically decomposes into a product of two-point functions, with a vertex factor V at each branch point.

The survival probability

Let $\theta_n(p) = \mathbb{P}_p(\exists x \in \mathbb{Z}^d : (0,0) \to (x,n)) \equiv \mathbb{P}_p(\mathbf{0} \to n)$.

Question: Behaviour of $\theta_n(p)$ as $n \to \infty$? Clearly $\theta_n(p) \downarrow \theta(p)$ as $n \uparrow$,

so $\theta_n(p) \to 0$ (exponentially fast) for $p < p_c$, and $\theta_n(p) \to \theta(p) > 0$ for $p > p_c$.

Since no percolation at p_c, $\theta_n(p_c) \to 0$ as $n \to \infty$. In what manner?

Prediction is $\theta_n(p_c) \approx n^{-1/\rho}$.

For critical Galton–Watson branching process with offspring distribution having mean 1 and variance σ^2,

$$\hat{\theta}_n \sim \frac{2}{\sigma^2 n}.$$

So the mean-field value of ρ is 1.

The survival probability: main result

Extinction probability: Define

$$\begin{aligned}\Delta\theta_n(p) &= \theta_n(p) - \theta_{n+1}(p)\\ &= \mathbb{P}_p((0,0) \longrightarrow n,\ (0,0) \not\longrightarrow n+1).\end{aligned}$$

Theorem. van der Hofstad – den Hollander – S 2007. For $d > 4$, $\exists L_0(d)$ such that, for $L \geq L_0$, as $n \longrightarrow \infty$,

$$\Delta\theta_n(p_c) = \frac{2}{AVn^2}\left[1 + O(n^{-1}\log n) + L^{-d}O(\delta_n)\right]$$

with

$$\delta_n = \begin{cases} n^{-(d-4)/2}\log n & (4 < d < 6)\\ n^{-1}\log^2 n & (d = 6)\\ n^{-1}\log n & (d > 6),\end{cases}$$

and thus, as $n \longrightarrow \infty$,

$$\theta_n(p_c) = \sum_{m=n}^{\infty}\Delta\theta_m(p_c) = \frac{2}{AVn}\left[1 + O(n^{-1}\log n) + L^{-d}O(\delta_n)\right].$$

The survival probability: constants

$$\theta_n(p_c) \sim \frac{2}{AVn}$$

The constants A, V are those seen before. In particular,

$$\sum_{x \in \mathbb{Z}^d} \tau_n(x) = A[1 + o(1)],$$

$$\sum_{x_1, x_2 \in \mathbb{Z}^d} \tau_{n_1,n_2}(x_1, x_2) = A^3 V (n_1 \wedge n_2)[1 + o(1)] \qquad \text{as } n_1 \wedge n_2 \longrightarrow \infty,$$

with $A = 1 + O(L^{-d})$ and $V = 1 + O(L^{-d})$ as $L \to \infty$.

Survival is rare but vigorous when it occurs

Theorem. For $d > 4$ and $L \geq L_0(d)$, conditional on survival to time n, $n^{-1}N_n$ converges weakly to an exponential random variable with parameter $\lambda = 2/(A^2V)$.

Proof. For $l \geq 1$,

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}[(n^{-1}N_n)^l | N_n > 0] &= \lim_{n\to\infty} \frac{1}{\theta_n}\frac{1}{n^l} \sum_{y_1,\ldots,y_l} \mathbb{P}((0,0) \to (y_j, n)) \\
&= \lim_{n\to\infty} \frac{AVn}{2}\frac{1}{n^l} \hat{\tau}^{(l+1)}_{n,\ldots,n}(\vec{0}) \\
&= (A^2V)^l 2^{-l} l!.
\end{aligned}$$

The statement that $\mathbb{E}[N_n] = \sum_x \tau_n(x) \to A$ can now be understood to correspond to the two statements

$$\mathbb{P}(N_n > 0) \sim \frac{2}{AVn}, \qquad \mathbb{E}[N_n | N_n > 0] \sim \frac{A^2V}{2}n.$$

In other words, clusters rarely survive to time n, but when they do, they are large.

Survival probability for critical branching process

Consider a critical branching process with offspring distribution q_m such that:

$$\text{mean } \mu = \sum_{m=0}^{\infty} m q_m = 1, \quad \text{variance } \sigma^2 = \sum_{m=0}^{\infty} m(m-1) q_m, \quad \text{survival prob. } \hat{\theta}_n.$$

Conditioning on number of offspring of initial particle that survive to time $n+1$ leads to:

$$\hat{\theta}_{n+1} = \hat{\theta}_n - \frac{\sigma^2}{2}\hat{\theta}_n^2 + O(\hat{\theta}_n^3).$$

Let $\hat{v}_n = 1/\hat{\theta}_n$, so that $\hat{v}_{n+1} = \hat{v}_n + \frac{1}{2}\sigma^2 + O(\hat{v}_n^{-1})$.

Easily conclude that $\hat{v}_n = \frac{1}{2}\sigma^2 n + O(\log n)$, so that

$$\hat{\theta}_n = \frac{2}{\sigma^2 n}[1 + O(n^{-1}\log n)].$$

Survival probability for oriented percolation

For any d, p, lace expansion for survival probability (a point-to-plane expansion) gives

$$\theta_n = \sum_{m=0}^{n-1} \pi_m p \theta_{n-1-m} - \sum_{m_1=1}^{\lfloor n/2 \rfloor} \sum_{m_2=m_1}^{n} \phi_{m_1,m_2} \theta_{n-m_1} \theta_{n-m_2} + e_n,$$

and suitable diagrammatic estimates are proved for π_m, ϕ_{m_1,m_2} and e_n, valid for $p = p_c$ (rescaled so $p_c \sim 1$), $d > 4$ and $L \geq L_0(d)$, e.g.,

(i) $\pi_0 = 1$, $\pi_1 = 0$, $|\pi_m| \leq C_\pi L^{-d} m^{-d/2}$ $(m \geq 2)$, and $p_c \sum_{m=0}^{\infty} \pi_m = 1$.

(ii) $\phi_{1,1} = \frac{1}{2}[1 + O(L^{-d})]$ and

$$|\phi_{m_1,m_2}| \leq C_\phi L^{-d} m_1^{-(d-2)/2} (m_2 - m_1)^{-(d-2)/2} \quad (m_2 \geq m_1 \geq 1,\ m_1 + m_2 \geq 3).$$

The recursion is then analysed via induction, with induction hypothesis on $v_j = 1/\theta_j$.

The incipient infinite cluster

Fix $p = p_c$. The IIC is a natural measure $\mathbb{P}_\infty$ such that $\mathbb{P}_\infty(|C(0,0)| = \infty) = 1$.

Easy to construct IIC on a tree: Kesten, Barlow-Kumagai.

IIC for ordinary percolation constructed by Kesten for $d = 2$:
Condition on $\{0 \longrightarrow \partial B_n\}$ at p_c and let $n \longrightarrow \infty$,
or condition on $\{0 \longrightarrow \infty\}$ at $p > p_c$ and let $p \downarrow p_c$.

Járai: several other constructions for $d = 2$.

van der Hofstad – Járai: construction of IIC for $d > 6$ (spread-out model).

The IIC for oriented percolation

Let $\tau_n(x) = \mathbb{P}_{p_c}((0,0) \longrightarrow (x,n))$ and $\tau_n = \sum_x \tau_n(x)$. Let

$$\mathbb{P}_n(E) = \frac{1}{\tau_n} \sum_x \mathbb{P}(E \cap \{(0,0) \longrightarrow (x,n)\}),$$

$$\mathbb{Q}_n(E) = \mathbb{P}_{p_c}(E \mid (0,0) \longrightarrow n).$$

Theorem. vdH–dH–S 02,07. For $d > 4$, $p = p_c$, $L \geq L_0(d)$,

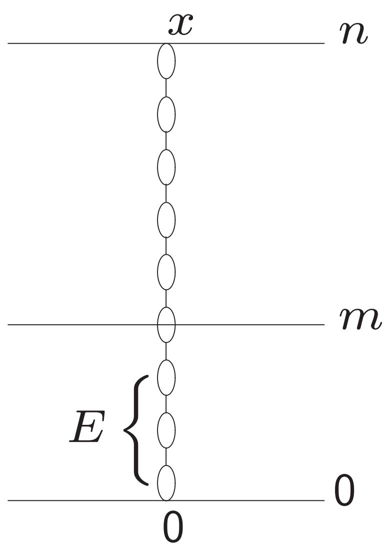
$$\mathbb{P}_\infty(E) = \lim_{n\to\infty} \mathbb{P}_n(E) \quad \text{exists}, \quad \mathbb{Q}_\infty(E) = \lim_{n\to\infty} \mathbb{Q}_n(E) \quad \text{exists}, \quad \mathbb{Q}_\infty = \mathbb{P}_\infty,$$

and $\mathbb{Q}_\infty((0,0) \longrightarrow \infty) = 1$.

Idea of proof of existence of $\mathbb{P}_\infty$

$$\mathbb{P}_n(E) = \frac{1}{\tau_n} \sum_{x \in \mathbb{Z}^d} \mathbb{P}(E \cap \{(0,0) \to (x,n)\}).$$

Regard a configuration contributing to $E \cap \{(0,0) \to (x,n)\}$ as a “string of sausages:”



where $E \in \mathcal{E}_m$, i.e., E depends only on bonds below level m.

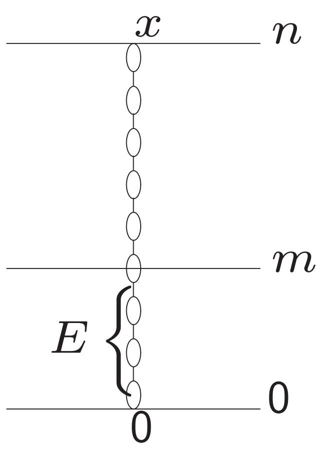
Idea of proof of existence of $\mathbb{P}_\infty$

A variant of the lace expansion, which to leading order treats as independent the sausages below and above m, gives

$$\mathbb{P}_n(E) = \frac{1}{\tau_n}\left[\sum_{l=m}^{n-1} \varphi_l(E)\tau_{n-l-1} + \varphi_n(E)\right]$$

with $|\varphi_l(E)| \leq C(l-m+1)^{-(d-2)/2}$ (for $d > 4$, L large). Can then take limit to get

$$\mathbb{P}_\infty(E) = \sum_{l=m}^{\infty} \varphi_l(E) \quad (E \in \mathcal{E}_m).$$



Existence of $\mathbb{Q}_\infty$ is similar but uses asymptotics of survival probability.

Geometry of IIC

Theorem. Let $N_n = \#\{y \in \mathbb{Z}^d : (0,0) \to (y,n)\}$. Under $\mathbb{P}_\infty$, $n^{-1}N_n$ converges weakly to a size-biased exponential random variable (density $\lambda^2 x e^{-\lambda x}$) with parameter $\lambda = 2/(A^2 V)$.

Proof. Explicit computation of moments as before.

Theorem. Let $B_n = \sum_{m=0}^n N_n$. Under $\mathbb{P}_\infty$, $n^{-2}B_n$ converges weakly to a nondegenerate random variable.

Proof. Tomorrow.

Theorem. IIC is 4-dimensional: Under $\mathbb{P}_\infty$, (in fact more can be said)

$$c_1 R^4 \leq \mathbb{E}_\infty\big[\#\{(y,m) \in C(0,0) : |y| \leq R\}\big] \leq c_2 R^4.$$

Proof. Analysis of the IIC two-point function

$$\rho_m(y) = \mathbb{P}_\infty((0,0) \to (y,m)) = \lim_{n\to\infty} \frac{1}{\tau_n} \sum_x \tau^{(3)}_{n,m}(x,y).$$