Random walks and the lace expansion III

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Abstract

We give an introduction to the lace expansion with emphasis on recent applications to:
1) the enumeration of self-avoiding walks, and
2) the analysis of random walks on the incipient infinite cluster for oriented percolation.

Preprint at `http://www.math.ubc.ca/~slade`.

More specifically:

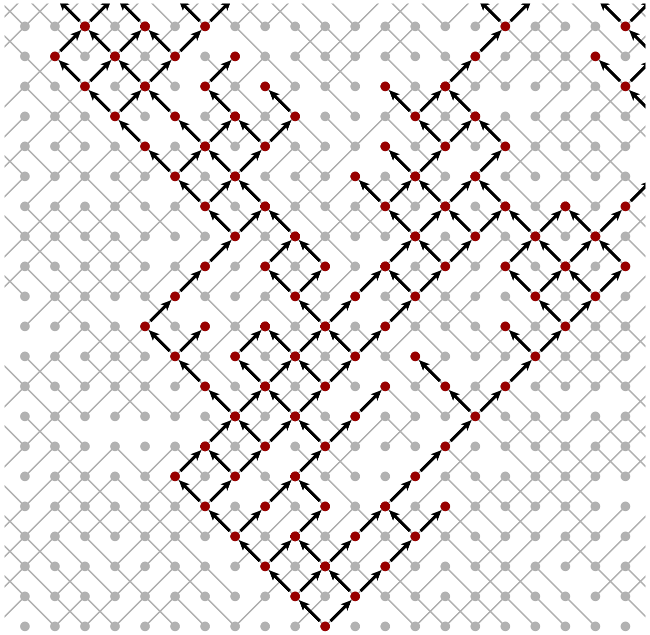
We prove estimates for exit times, transition probabilities, and the range of random walk on the IIC for spread-out oriented percolation on $\mathbb{Z}^d \times \mathbb{Z}_+$ for $d > 6$. These estimates show that the IIC has spectral dimension $\frac{4}{3}$.

Joint work with Martin Barlow, Antal Járai, Takashi Kumagai.

Spread-out oriented percolation

Bond percolation on the directed graph with vertices (x, n), $x \in \mathbb{Z}^d$, $n = 0, 1, 2, \ldots$ and directed bonds $((x, n), (y, n+1))$ with $\|x - y\| \le L$ (later: L large).

Bonds are independently "occupied" with probability p and otherwise "vacant."



$p = 0.7 > p_c \approx 0.645$

Phase transition for oriented percolation

We say that $(0,0) \longrightarrow (x,n)$ if there is an occupied oriented path from $(0,0)$ to (x,n), and let

$$C(0,0) = \{(x,n) : (0,0) \longrightarrow (x,n)\}.$$

Let $\theta(p) = \mathbb{P}_p(|C(0,0)| = \infty)$. Phase transition: $\exists\, p_c = p_c(d,L) \in (0,1)$ s.t.

$$\theta(p) = 0 \quad \text{if } p < p_c,$$

$$\theta(p) > 0 \quad \text{if } p > p_c.$$

For $p = p_c$: Bezuidenhout–Grimmett (1990) proved $\theta(p_c) = 0$.

Critical behaviour for $p \approx p_c$ well understood for $d > 4$ (when $L \gg 1$).

The incipient infinite cluster

Let $\tau_n(x) = \mathbb{P}_{p_c}((0,0) \to (x,n))$ and $\tau_n = \sum_x \tau_n(x)$. Let

$$\mathbb{P}_n(E) = \frac{1}{\tau_n} \sum_x \mathbb{P}(E \cap \{(0,0) \to (x,n)\}),$$

$$\mathbb{Q}_n(E) = \mathbb{P}_{p_c}(E \mid (0,0) \to n).$$

Theorem (vdH–dH–S 02,07). For $d > 4$, $p = p_c$, $L \geq L_0(d)$,

$$\mathbb{P}_\infty(E) = \lim_{n\to\infty} \mathbb{P}_n(E) \quad \text{exists}, \quad \mathbb{Q}_\infty(E) = \lim_{n\to\infty} \mathbb{Q}_n(E) \quad \text{exists}, \quad \mathbb{Q}_\infty = \mathbb{P}_\infty,$$

and $\mathbb{Q}_\infty((0,0) \to \infty) = 1$.

IIC for unoriented percolation on $\mathbb{Z}^d$: Kesten, Járai for $d = 2$; van der Hofstad and Járai for $d > 6$ (spread-out).
Much easier to construct IIC on a tree: Kesten, Barlow-Kumagai.

Random walk on the IIC

Goal of this work: study the long-time behaviour of random walk on the IIC.

Much recent work on random walk on supercritical percolation clusters:
Barlow, Berger–Biskup, Mathieu–Piatnitski, Sidoravicius–Sznitman, . . .

Kesten proved that random walk on the IIC for 2-dimensional unoriented percolation is subdiffusive, i.e., $\{n^{-1/2+\epsilon}X_n\}_{n\geq 1}$ is tight.

Random walk on the IIC for critical Galton–Watson trees is well understood: Kesten, Barlow–Kumagai.

Infinite random graphs

Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space carrying a family of random graphs $\Gamma(\omega) = (G(\omega), E(\omega))$, which are infinite, connected, locally-finite, with distinguished vertex o.

Let $B(R)$ be the set of vertices within distance R of o. Let μ_x denote the degree of a vertex x. Extend μ to a measure on G. Let $V(R) = \mu(B(R))$.

Define a quadratic form $\mathcal{E}$ by

$$\mathcal{E}(f, g) = \tfrac{1}{2} \sum_{x,y \in G: \{x,y\} \in E} (f(x) - f(y))(g(x) - g(y)).$$

Then $\mathcal{E}(f, f)$ is the energy dissipation when the vertices of G are at a potential f and edges have unit resistance. Set $H^2 = \{f : G \to \mathbb{R} : \mathcal{E}(f, f) < \infty\}$. Let A, B be disjoint subsets of G. The effective resistance between A and B is defined by:

$$R_{\text{eff}}(A, B)^{-1} = \inf\{\mathcal{E}(f, f) : f \in H^2, f|_A = 1, f|_B = 0\}.$$

Assumption on the random environment

We assume that $V(R) \approx R^2$ and $R_{\text{eff}}(o, B(R)^c) \approx R$, namely:

Assumption. Suppose that:

Volume: $\mathbb{E}V(R) \leq c_2 R^2$ and $\mathbb{E}[\frac{1}{V(R)}] \leq c_3 R^{-2}$.

Effective resistance: $\mathbb{P}(R_{\text{eff}}(o, B(R)^c) \leq \epsilon R) \leq c_1 \epsilon^q$ for some $q > 0$.

Random walk on an infinite random graph

Given a random environment $\Gamma(\omega) = (G(\omega), E(\omega))$, consider simple random walk with transition probabilities: $P_\omega(X_{n+1} = y | X_n = x) = \mu_x^{-1}$ for $\{x, y\} \in E$.

Interested in:

- $\tau_R = \min\{n : d(o, X_n) = R\}$,
- $p_{2n}^\omega(x, x) = \frac{1}{\mu_x} P_\omega(X_{2n} = x | X_0 = x)$ (for $x \in G$),
  spectral dimension: $p_{2n}^\omega(x, x) \approx n^{-d_s/2}$,
- $W_n =$ the number of points visited by time n.

Random walk on random graph: Results

Theorem (BJKS 2006). Suppose the assumption holds. Then there is a set Ω_0 of graphs with $\mathbb{P}(\Omega_0) = 1$, and $\alpha_1, \alpha_2 > 0$ such that for each $\omega \in \Omega_0$ and for each $x \in G(\omega)$:

$$(\log R)^{-\alpha_1} R^3 \leq E_\omega^x \tau_R \leq (\log R)^{\alpha_1} R^3 \quad \forall R \geq R_x(\omega),$$

$$(\log n)^{-\alpha_2} n^{-2/3} \leq p_{2n}^\omega(x,x) \leq (\log n)^{\alpha_2} n^{-2/3} \quad \forall n \geq N_x(\omega),$$

$$\lim_{n\to\infty} \frac{\log \mu(W_n)}{\log n} = \frac{2}{3} \quad P_\omega^x\text{-a.s.}$$

In particular,

$$d_s(\Gamma(\omega)) = \frac{4}{3}.$$

Same subdiffusive power laws as those on the IIC for a tree (Kesten 1986, Barlow–Kumagai 2006).

Generalisation: Kumagai–Misumi 2007.

Two examples of (non-random) graphs

Two examples showing relevance of the volume $V(R)$ and the effective resistance R_{eff}:

Example 1. Big volume slows down the walk. Decorate a path of length R by adding a complete graph on S vertices at each vertex along the path (except the last couple). To travel distance R from the root, the RW must reach the other endpoint of the path. It typically enters each complete graph, takes on average S steps inside the complete graph before finding its path vertex again, and it must revisit that vertex on average S times before moving along the path. On average, it will spend time R^2 in the path, for a total time of $R^2S^2 = RV(R)$.

Example 2. Low resistance speeds up the walk. Consider a graph consisting of R disjoint paths of length R, joined at the root. Then $V(R) = R^2$. RW will travel distance R in average time R^2. The walk is faster due to low resistance: $R_{\mathrm{eff}}(0, B(R)^c) = R/R = 1$.

In each example, time is $R_{\mathrm{eff}}(0, B(R)^c)V(R)$.

Sample argument for $E^x \tau_R$: Upper bound in non-random environment

Fix $\Gamma = (G, E)$ and $x = o \in G$. Let $B = B(R)$.

Upper bound: $E^o \tau_R \le R_{\text{eff}}(o, B^c) V(R)$.

Proof.

$$
\begin{aligned}
E^o \tau_R &= \sum_{y \in B} g_B(o, y) \mu_y \\
&\le g_B(o, o) \sum_{y \in B} \mu_y = g_B(o, o) V(R) \\
&= R_{\text{eff}}(o, B^c) V(R).
\end{aligned}
$$

With a bit more work: **Lower bound:** Suppose that $R_{\text{eff}}(o, B^c) \ge \epsilon R$. Then

$$
E^o \tau_R \ge \frac{1}{4} R_{\text{eff}}(o, B^c) V(\tfrac{1}{2} \epsilon R).
$$

Examples where the assumption is satisfied

1. The IIC for Bin$(m, \frac{1}{m})$ Galton–Watson tree (Barlow-Kumagai 06, Kesten 86).
2. Invasion percolation cluster on a regular tree (Angel-Goodman-Hollander-S 06).
   The IPC is stochastically dominated by the IIC. Since $R_{\rm eff}(o, B(R)^c)$ increases when edges are removed from the graph, the lower bound on $R_{\rm eff}$ for the IPC follows from the known lower bound for the IIC. The upper bound on the volume for the IPC also follows from stochastic domination and the known volume behaviour of the IIC.
   AGHS: The lower bound on the volume holds for the IPC.
3. The IIC for spread-out oriented percolation for $d > 6$ and $L \gg 1$ (BJKS 06).
   This verifies the Alexander–Orbach conjecture (which is now believed to be false in low dimensions).
   We concentrate on this example henceforth.

Volume estimates for the IIC: Upper bound

Let

$$Z_R = aR^{-2}V(R),$$

where a is a certain constant. Let Y_t have distribution the canonical measure of super-Brownian motion conditioned to survive for all time (van der Hofstad 06). Let

$$Z = \int_0^1 dt\ Y_t(\mathbb{R}^d).$$

Proposition. Let $d > 4$ and $L \geq L_0$. Under the IIC measure, Z_R converges in distribution to Z. Moreover, all moments converge, i.e., $\mathbb{E}_\infty Z_R^l \longrightarrow \mathbb{E} Z^l$ $\forall l$. In particular,

$$c_1 R^2 \leq \mathbb{E}_\infty V(R) \leq c_2 R^2.$$

Volume estimates: Idea of proof of upper bound

Let $B(R) = \{(y,n) \in \mathcal{C} : 0 \le n < R\}$. Let

$$\tilde{Z}_R = \tilde{a} R^{-2} |B(R)| = \tilde{a} R^{-2} \sum_{n=0}^{R-1} \sum_{y \in \mathbb{Z}^d} I[(0,0) \to (y,n)].$$

It suffices to show that $\lim_{R\to\infty} \mathbb{E}_\infty \tilde{Z}_R^l = \mathbb{E} Z^l$ for all $l \ge 1$.

For $l \ge 1$, $\vec{m} = (m_1, \dots, m_l)$ and $y_i \in \mathbb{Z}^d$, let

$$\hat{\rho}^{(l+1)}_{\vec{m}} = \sum_{y_1,\dots,y_l \in \mathbb{Z}^d} \mathbb{P}_\infty((0,0) \to (y_i, m_i) \text{ for all } i = 1, \dots, l).$$

Let $\vec{t} = (t_1, \dots, t_l) \in (0,1]^l$. HHS proved that

$$\lim_{m\to\infty} \left(\frac{\tilde{a}}{m}\right)^l \hat{\rho}^{(l+1)}_{m\vec{t}} = \hat{M}^{(l+1)}_{1,\vec{t}} = \mathbb{N}\big(X_1(\mathbb{R}^d) X_{t_1}(\mathbb{R}^d) \cdots X_{t_l}(\mathbb{R}^d)\big);$$

RHS is the $(l+1)^{\mathrm{st}}$ moment of the canonical measure $\mathbb{N}$ of super-Brownian motion X_t.

Volume estimates: Idea of proof of upper bound

But this implies, for $l \geq 1$, that

$$\mathbb{E}_\infty \tilde{Z}_R^l = \frac{\tilde{a}^l}{R^{2l}} \sum_{n_1=0}^{R-1} \cdots \sum_{n_l=0}^{R-1} \hat{\rho}^{(l+1)}_{n_1,\ldots,n_l}$$

$$= \frac{1}{R} \sum_{n_1=0}^{R-1} \cdots \frac{1}{R} \sum_{n_l=0}^{R-1} \frac{\tilde{a}^l}{R^l} \hat{\rho}^{(l+1)}_{\vec{t}R} \qquad \text{[here } \vec{t} = (n_1 R^{-1}, \ldots, n_l R^{-1})\text{]}$$

$$\to \int_0^1 dt_1 \cdots \int_0^1 dt_l \hat{M}^{(l+1)}_{1,\vec{t}} = \mathbb{E} Z^l.$$

Volume estimates for the IIC: Lower bound

Proposition. Let $d > 4$, $L \geq L_0$, $R \geq 1$.

$$\mathbb{Q}_\infty(V(R)R^{-2} < \lambda) \leq c_1 e^{-c_2/\sqrt{\lambda}}.$$

This estimate is of the same form as the corresponding estimate for the IIC for critical Galton–Watson trees.

Omit details here.

Resistance estimate for the IIC: cut set

Say that $(x,n),(x',n')$ in the IIC are *RW-connected*, if there is a path, not necessarily oriented, in $\mathcal{C}$ from (x,n) to (x',n').

Let $D(n)$ denote the *cut set* at level n: those bonds $e=((w,n-1),(x,n))$ which must be cut in order to RW-disconnect $(0,0)$ from level R.

This is well-defined, and consists of those e whose top is RW-connected to level R without going below level n:

all such edges need to be cut in order to RW-disconnect 0 from level R, and

cutting all such edges RW-disconnects 0 from $B(R)^c$ (since for any RW-path from 0 to $B(R)^c$ the last crossing of level n occurs at such an edge).

Resistance estimate for the IIC

Proposition. Assume $d > 6$. There exists $L_1 = L_1(d) \geq L_0(d)$ such that for $L \geq L_1$,

$$\mathbb{E}_\infty(|D(n)|) \leq c_1, \qquad 0 < n \leq 3R/4.$$

The proposition is the only place where we need $d > 6$ rather than $d > 4$.

Lemma. For oriented percolation in any dimension $d \geq 1$,

$$R_{\text{eff}}(0, \partial B(R)) \geq \sum_{n=1}^{3R/4} \frac{1}{|D(n)|}.$$

With a little extra work, this gives the assumed resistance bound.

Role of $d > 6$ in the resistance estimate

Write $\boldsymbol{w} = (w, n-1)$, $\boldsymbol{x} = (x, n)$, $\boldsymbol{y} = (y, N)$. Then

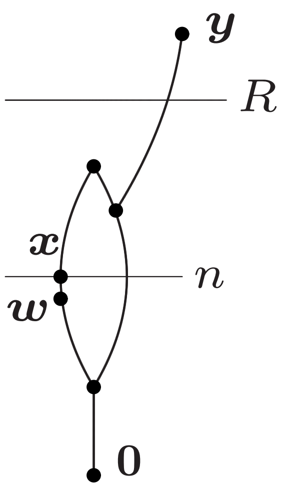
$$\begin{aligned}
\mathbb{E}_\infty |D(n)| &= \sum_{w,x \in \mathbb{Z}^d} \mathbb{P}_\infty \left[(\boldsymbol{w}, \boldsymbol{x}) \in D(n) \right] \\
&= \lim_{N \to \infty} \frac{1}{\tau_N} \sum_{w,x,y \in \mathbb{Z}^d} \mathbb{P}_{p_c} \left[(\boldsymbol{w}, \boldsymbol{x}) \in D(n),\ \mathbf{0} \to \boldsymbol{y} \right] \\
&= \frac{1}{A} \lim_{N \to \infty} \sum_{w,x,y \in \mathbb{Z}^d} \mathbb{P}_{p_c} \left[(\boldsymbol{w}, \boldsymbol{x}) \in D(n),\ \mathbf{0} \to \boldsymbol{y} \right],
\end{aligned}$$

because

$$\lim_{N \to \infty} \tau_N = A.$$

One configuration in the event $\{(\boldsymbol{w}, \boldsymbol{x}) \in D(n),\ \mathbf{0} \to \boldsymbol{y}\}$:

$$\sum_{w,x,y}$$



The bounds $\|\tau_m\|_1 \leq 2A$ and $\|\tau_m\|_\infty \leq C m^{-d/2}$ give the upper bound:

$$c \sum_{l=n}^{\infty} \sum_{k=n}^{l} \sum_{j=0}^{n} (l - j + 1)^{-d/2} \leq c \sum_{l=n}^{\infty} \sum_{k=n}^{l} (l - n + 1)^{(2-d)/2}$$

$$\leq c \sum_{l=n}^{\infty} (l - n + 1)^{(4-d)/2} = c \sum_{m=1}^{\infty} m^{(4-d)/2},$$

which is finite only for $d > 6$. Complete proof estimates the contributions due also to more complex zigzag random walk paths.

$d > 6$ vs $d > 4$?

The upper critical dimension for oriented percolation is 4, not 6.

Consider the incipient infinite branching random walk (IIBRW), obtained as the limit as $n \to \infty$ of critical branching random walk (say binomial offspring distribution) conditioned to survive to at least n generations. Interpret IIBRW as a random infinite subgraph of $\mathbb{Z}^d \times \mathbb{Z}_+$, and consider simple random walk on the IIBRW.

Volume estimates: this part of the Assumption holds for the IIBRW for $d > 4$.

Effective resistance: $R_{\text{eff}}(0, B(R)^c)$ for the IIBRW is lower than it is for the IIC on a tree, due to cycles in the IIBRW.

Open problem: Determine the asymptotic behaviour of $R_{\text{eff}}(0, B(R)^c)$ for the IIBRW. Does $d > 4$ suffice for the Assumption, or is there different behaviour for $4 < d \leq 6$ and $d > 6$?