Spectral Properties of Saddle Point Linear Systems and Relations to Iterative Solvers
Part III: Nonsymmetric Preconditioners

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Outline of the 3-hour Presentation

- Schematic presentation of certain algebraic preconditioners (First day)
- Iterative solvers. Some (hopefully) helpful considerations... (Yesterday)
- Spectral analysis of nonsymmetric preconditioners (Today)

Why should we go nonsymmetric?

- Good (complex) clustering may be better than just “symmetry”
- More freedom in selecting a preconditioner
- If only few iterations, costs are low

Only for really tough problems

An example. The 3D magnetostatic problem.

(3D) Maxwell equations: $\operatorname{div}\mathbf{B} = 0 \quad \operatorname{curl}\mathbf{H} = \mathbf{J}$

Constitutive law

$$\mathbf{B} = \mu\mathbf{H}$$

$\mathbf{B}$ *displacement field,* $\mathbf{H}$ *magnetic field,* μ *magn.permeability*

Constrained quadratic programming formulation

$$\Downarrow$$

$$\min \frac{1}{2}\int_\Omega \mu^{-1}\,|\mathbf{B} - \mu\mathbf{H}|^2\,dx$$

$$\text{with} \quad \begin{array}{ll} \mathbf{B}\cdot\mathbf{n} = f_B & \text{on } \Gamma_B \\ \mathbf{H}\wedge\mathbf{n} = \mathbf{f}_H & \text{on } \Gamma_H \end{array} \quad \text{and} \quad \begin{array}{l} \operatorname{div}\mathbf{B} = 0 \\ \operatorname{curl}\mathbf{H} = \mathbf{J} \end{array}$$

⋆ *Boundary conditions are enforced*
⋆ *Constraints: Lagrange multipliers*

Triangular preconditioner

$$\mathcal{M} = \begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} \ A \text{ spd}, \quad \mathcal{P} = \begin{bmatrix} \widetilde{A} & B^T \\ 0 & -\widetilde{C} \end{bmatrix}, \quad \begin{array}{l} \widetilde{A} \approx A, \\ \widetilde{C} \approx BA^{-1}B^T + C \end{array}$$

Spectrum of $\mathcal{M}\mathcal{P}^{-1}$:

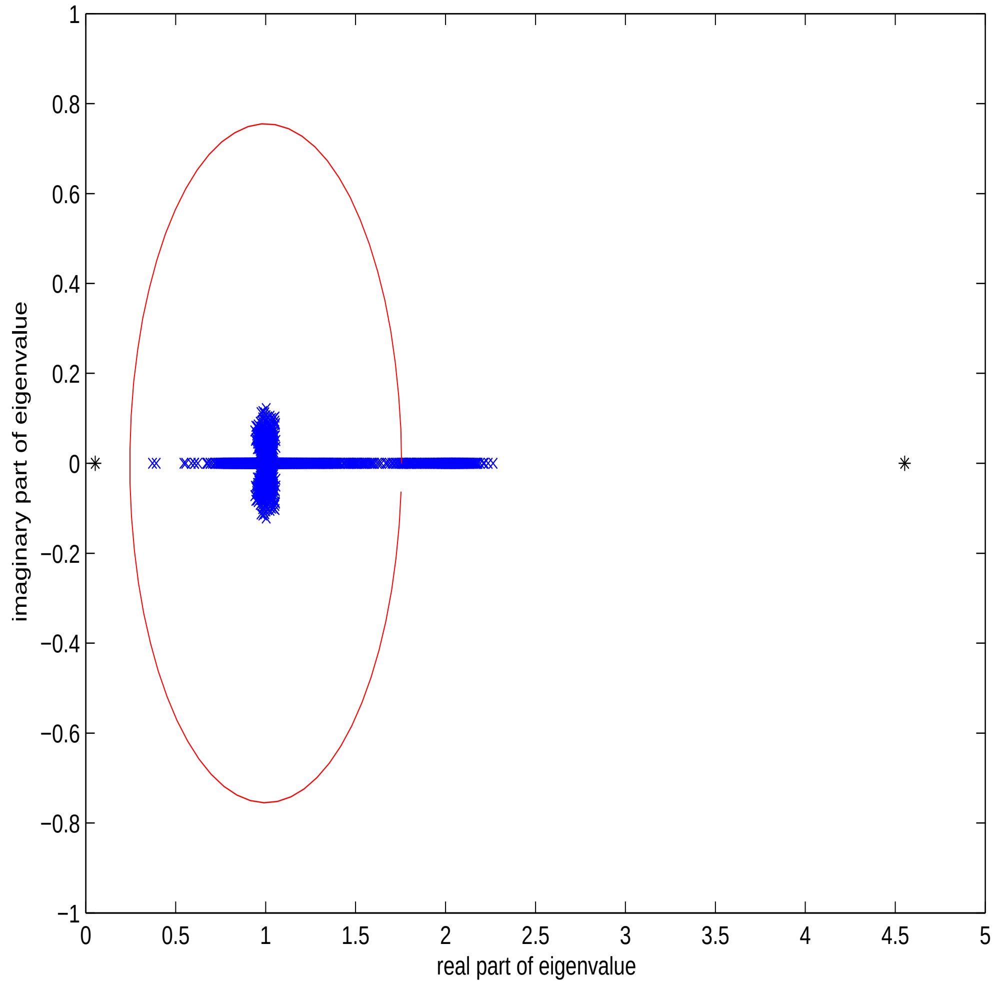
small complex cluster around 1 $\cup$ real interval :

More precisely: $\theta \in \Lambda(\mathcal{M}\mathcal{P}^{-1})$

$$\Im(\theta) \neq 0 \quad \Rightarrow |\theta - 1| \leq \sqrt{1 - \lambda_{\min}(A\widetilde{A}^{-1})} \qquad \text{(if } 1 - \lambda_{\min}(A\widetilde{A}^{-1}) \geq 0\text{)}$$

$$\Im(\theta) = 0 \quad \Rightarrow \theta \in [\chi_1, \chi_2] \text{ with } 1 \in [\chi_1, \chi_2]$$

An example. The 3D magnetostatic problem.



size: 2208. $\widetilde{A} = \texttt{cholinc}(A, 10^{-2})$, $\widetilde{C} = \texttt{cholinc}(C, 10^{-2})$

An example. 3D magnetostatic problem.

A spd, B not full rank, C s.t. $BB^T + C$ is spd

Problem dim.	(its) GMRES Block.Tr.	E-time (its) CGstab Block.Tr.	E-time (its) QMRs Block.Tr.	E-time (its) QMRs Block.Diag.
2208	(17)	0.59 (24)	0.46 (18)	0.88 (40)
8622	(30)	5.31 (44)	4.06 (31)	6.85 (64)
22675	(50)	26.08 (84)	18.57 (53)	32.85 (115)

$\widehat{A}$: ICT$(A, 10^{-2}, 2)$. $\widehat{C}$: ICT$(BB^T + C, 10^{-4}, 5)$

tol $= 10^{-8}$

More to show? An open problem

$$\begin{bmatrix} A & B^T \\ 0 & -C \end{bmatrix} \quad \Rightarrow \quad \mathcal{P} = \begin{bmatrix} \widetilde{A} & B^T \\ 0 & -\widetilde{C} \end{bmatrix}$$

⋆ Pretty clear understandanding of “perturbed” spectrum

Eigenvectors?

$$\|r_m^{gmres}\| \le \|r_0\|\kappa(X) \min_{p_m(0)=1} \max_{j=1,\ldots,n+m} \|p_m(\lambda_j)\|$$

X eigenvector matrix of $\mathcal{M}\mathcal{P}^{-1}$, $\lambda_j \in \sigma(\mathcal{M}\mathcal{P}^{-1})$

Indefinite (Constraint) Preconditioner

$$\mathcal{M}\mathcal{P}^{-1} \qquad \mathcal{P} = \begin{bmatrix} \widetilde{A} & B^T \\ B & O \end{bmatrix}, \qquad C = O,\ \widetilde{A} = I$$

Remark: $Bx_k = 0$ for all iterates x_k (constraint)

Indefinite (Constraint) Preconditioner

$$\mathcal{MP}^{-1} \qquad \mathcal{P} = \begin{bmatrix} \widetilde{A} & B^T \\ B & O \end{bmatrix}, \qquad C = O,\ \widetilde{A} = I$$

Remark: $Bx_k = 0$ for all iterates x_k (constraint)

$\Pi = B^T(BB^T)^{-1}B$ orth. projector

$$\mathcal{MP}^{-1} = \begin{bmatrix} (A-I)(I-\Pi)+\Pi & \star \\ O & I_m \end{bmatrix}$$

$\lambda \neq 0$ eigs of $\mathcal{MP}^{-1}$: $\qquad \lambda \in \mathbb{R}^+, \qquad \lambda \in \{1\} \cup \sigma(A(I-\Pi)+\Pi)$

$\mathcal{MP}^{-1}$ has nontrivial Jordan blocks

Canonical Form ($C = O$)

$$\mathcal{P}^{-1}\mathcal{M} = \mathcal{X}^{-1} \begin{bmatrix} \Theta & & \\ & I_m & I_m \\ & & I_m \end{bmatrix} \mathcal{X}$$

with

$$\mathcal{X} = \begin{bmatrix} I & B^T \\ O & (BB^T)^{\frac{1}{2}} \end{bmatrix} \left[\begin{array}{c|c|c} X_0 & B_0^T & (A-I)^{-1}B_0^T \\ B_0X_0 & I_m & O \end{array} \right]$$

where

- $B_0 = (BB^T)^{-\frac{1}{2}}B$ orth. rows
- X_0 eigenvec's of $(A-I)(I-\Pi)$ corresponding to nonzero eigs Θ:

$$(A-I)(I-\Pi)X_0 = X_0\Theta$$

Bound for the GMRES residual

$$\mathcal{J} = \begin{bmatrix} \Theta & & \\ & I_m & I_m \\ & & I_m \end{bmatrix}$$

Then

$$\|r_m\| \leq \|r_0\| \kappa(\mathcal{X}) \min_{p_m(0)=1} \|p_m(\mathcal{J})\|$$

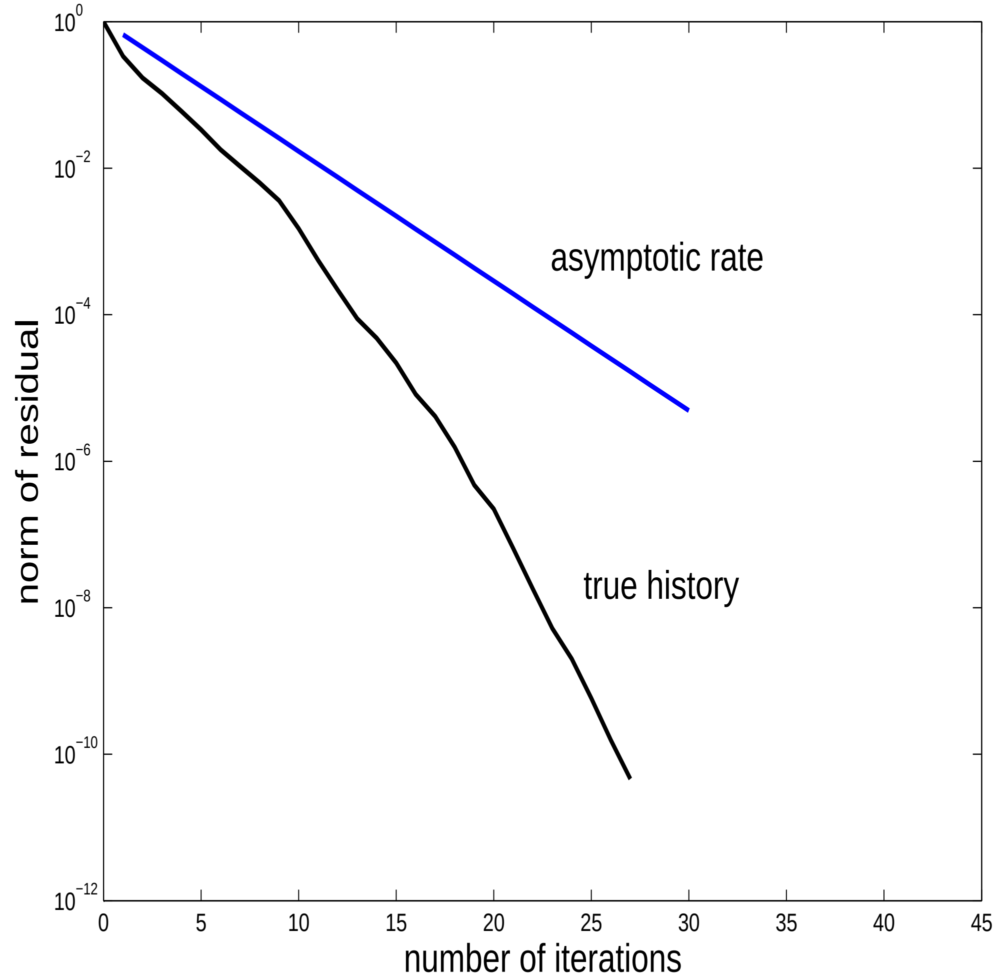
If 1 is inside the spectral interval,

$$\min_{p_m(0)=1} \|p_m(\mathcal{J})\| \approx \min_{p_m(0)=1} \max_{j=1,\ldots,n+m} |p_m(\lambda_j)| \leq 2 \left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1} \right)^m$$

with $\kappa = \frac{\lambda_{\max}}{\lambda_{\min}}$

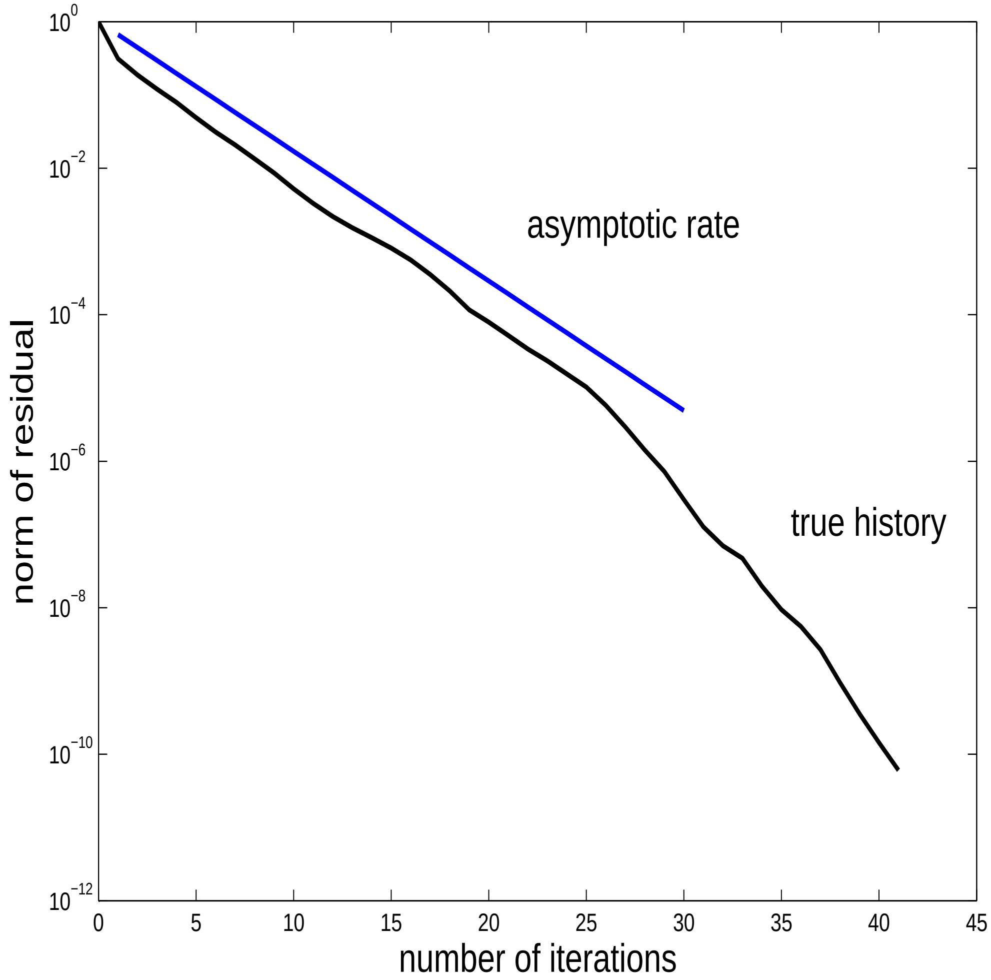
CG bound for the spectrum

Stokes problem with mixed b.c. on the unit square



$$\kappa(\mathcal{X}) = 9.4 \cdot 10^3$$

Stokes problem with mixed b.c. on the unit square



Random right-hand side (first block) $\kappa(\mathcal{X}) = 9.4 \cdot 10^3$

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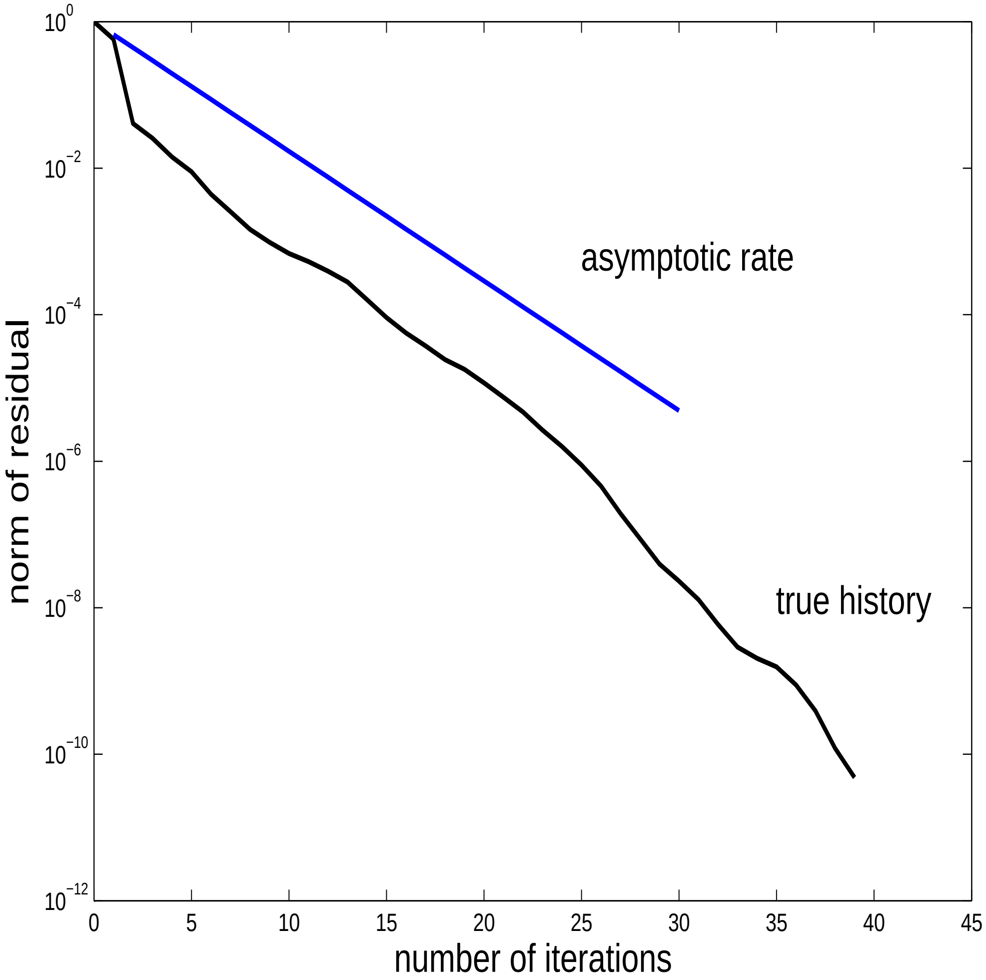
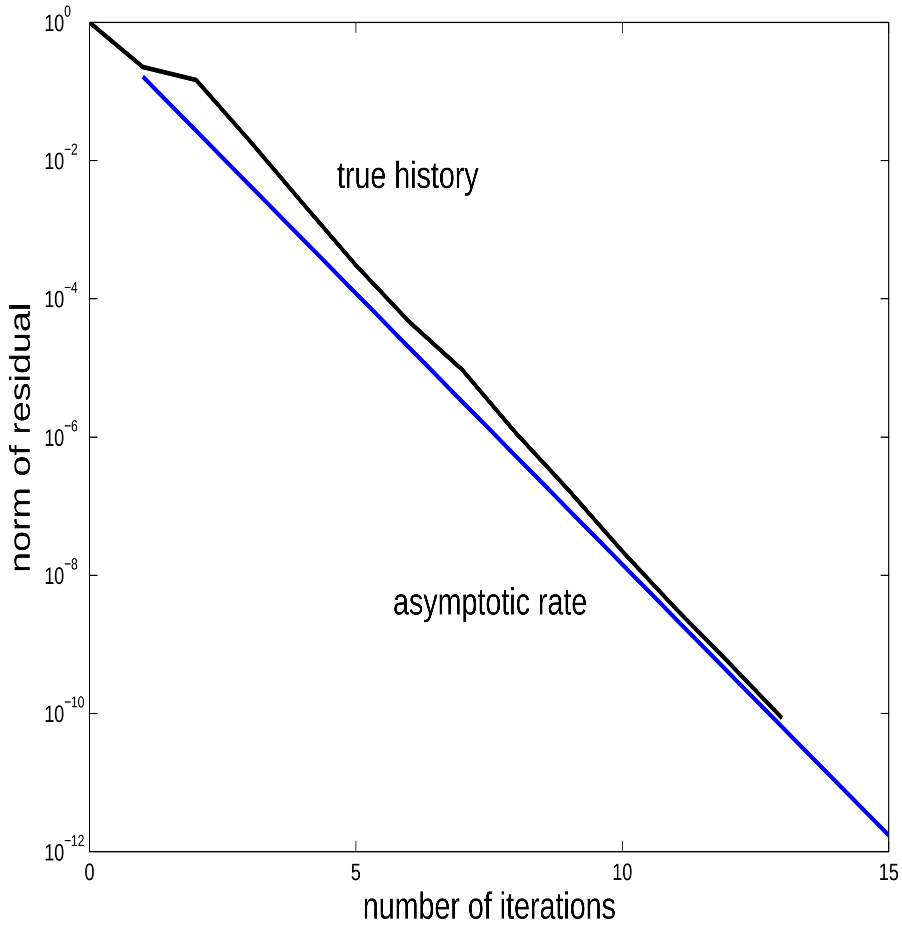


Random right-hand side (both blocks)

$\kappa(\mathcal{X}) = 9.4 \cdot 10^3$

The example from magnetostatic



$$\kappa(\mathcal{X}) = 2.6 \cdot 10^{9}$$

Dimension reduction

$$\begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

In this case, GMRES on $\mathcal{M}\mathcal{P}^{-1}$ satisfies:

$$\|r_m^{gmres}\| = \min_{p_m(0)=1} \|p_m((A-I)\Pi + \Pi)f)\|$$

where $\Pi = B^T(BB^T + C)^{-1}B$

From this,

$$\|r_m^{gmres}\| \leq \|r_0\|\zeta \left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^m$$

$\kappa = \lambda_{\max}/\lambda_{\min}$

ζ depends on $\|A\|, \|(A-I)^{-1}\|$

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$\kappa = \lambda_{\max}/\lambda_{\min}$ ζ depends on $\|A\|, \|(A-I)^{-1}\|$

⋆ Nonsym Lanczos-type methods reduce to PCG

A feasible preconditioner

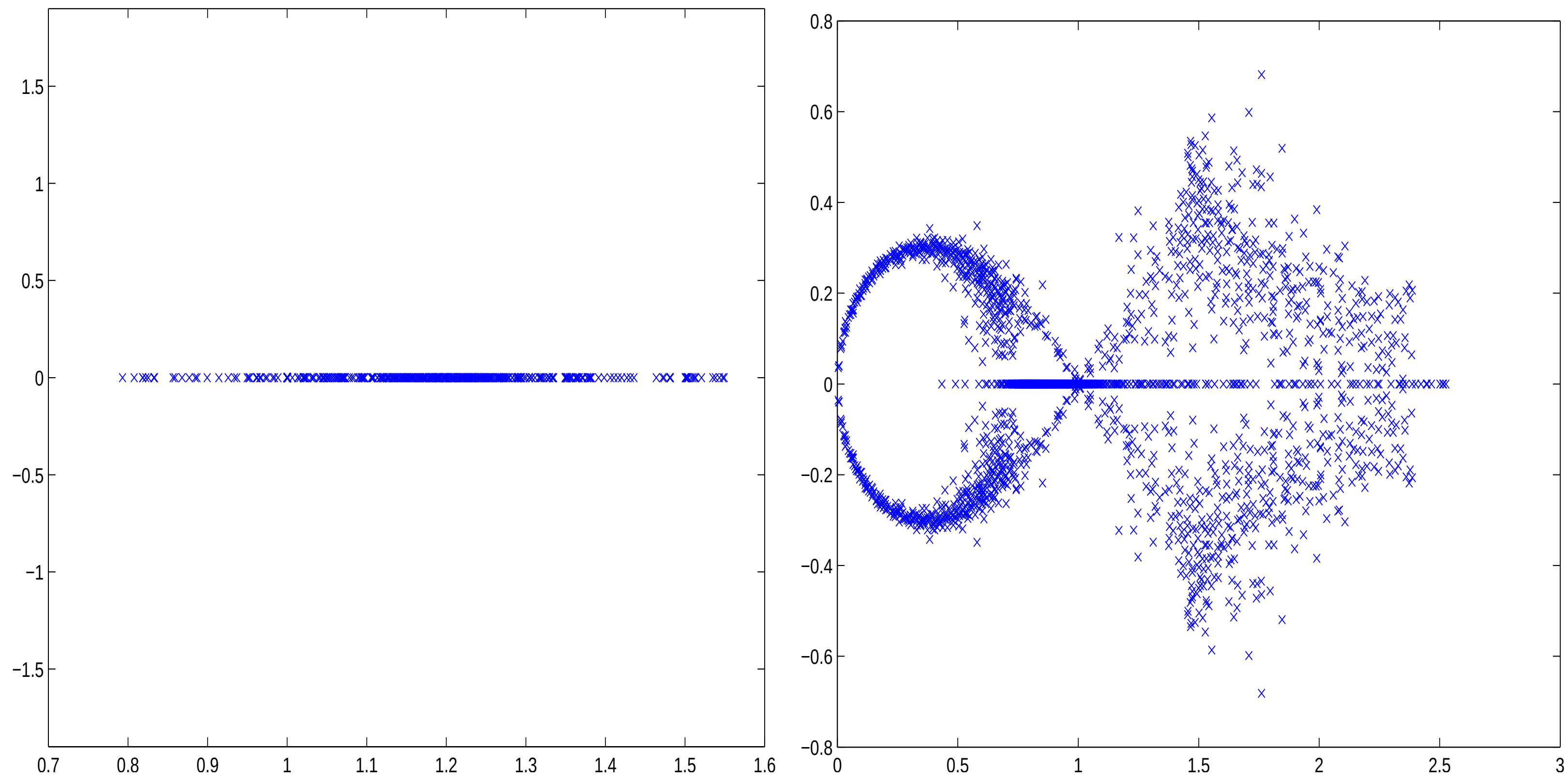
$$\mathcal{P} = \begin{bmatrix} I & B^T \\ B & -C \end{bmatrix}$$

$$\mathcal{P} = \begin{bmatrix} I & O \\ B & I \end{bmatrix} \begin{bmatrix} I & O \\ O & -(\mathbf{B}\mathbf{B}^{\mathbf{T}} + \mathbf{C}) \end{bmatrix} \begin{bmatrix} I & B^T \\ O & I \end{bmatrix}$$

Replace $BB^T + C$ with a cheaper approximation S

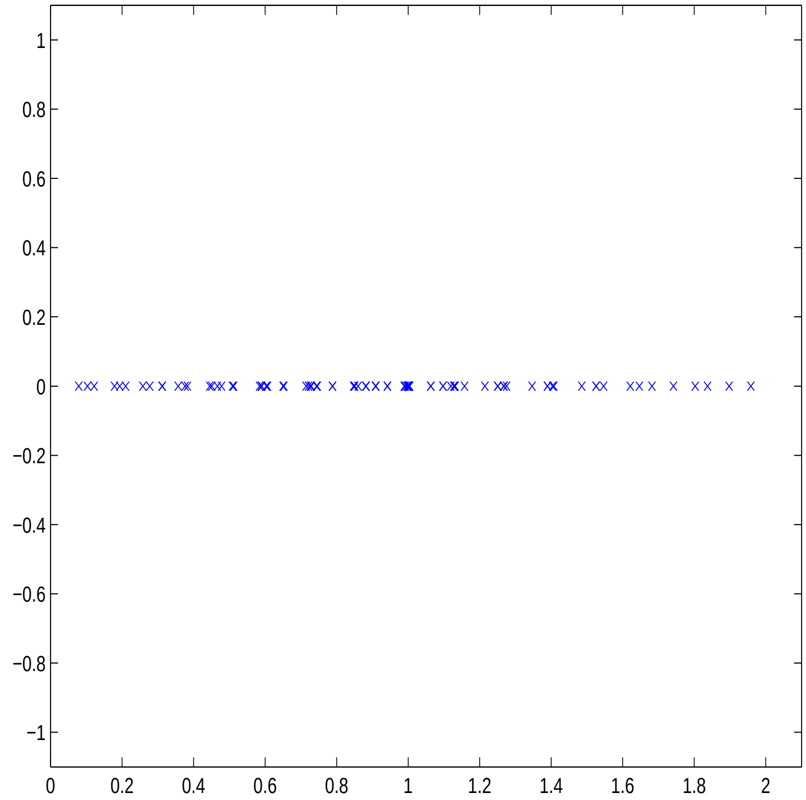
Question: How much does S affect the preconditioner ?

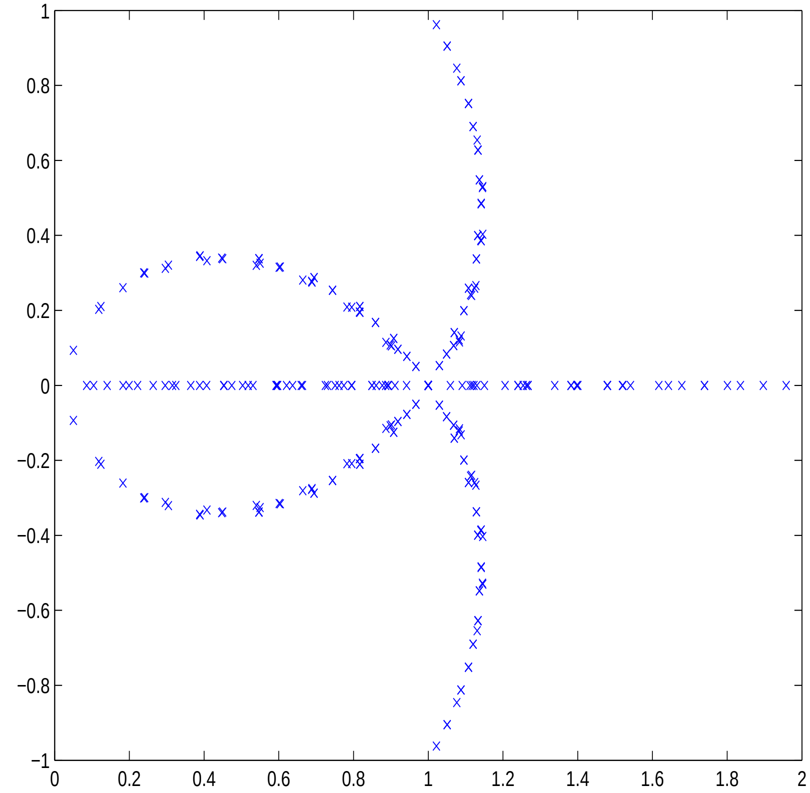
Spectrum of perturbed matrix. 2D Magnetostatic problem



$(C = 0)$, $\|BB^T - S\|_\infty \approx 0.5\|BB^T\|_\infty$ $\quad\quad$ $\widetilde{A} = I,\ S = \mathrm{diag}(BB^T)$

Spectrum of perturbed matrix. Stokes problem





$\|BB^T - S\|_\infty \approx 0.5\|BB^T\|_\infty \qquad \widetilde{A} = \mathrm{diag}(A),\; S = \mathrm{diag}(B\widetilde{A}^{-1}B^T)$

Eigenvalue bounds

$C = 0$, A spd. $S \approx BB^T$

Let $G = B(2I - A)B^T S^{-1}$ (scale A so that $G \geq 0$)

$\star$ If $\Im(\lambda) \neq 0$ then

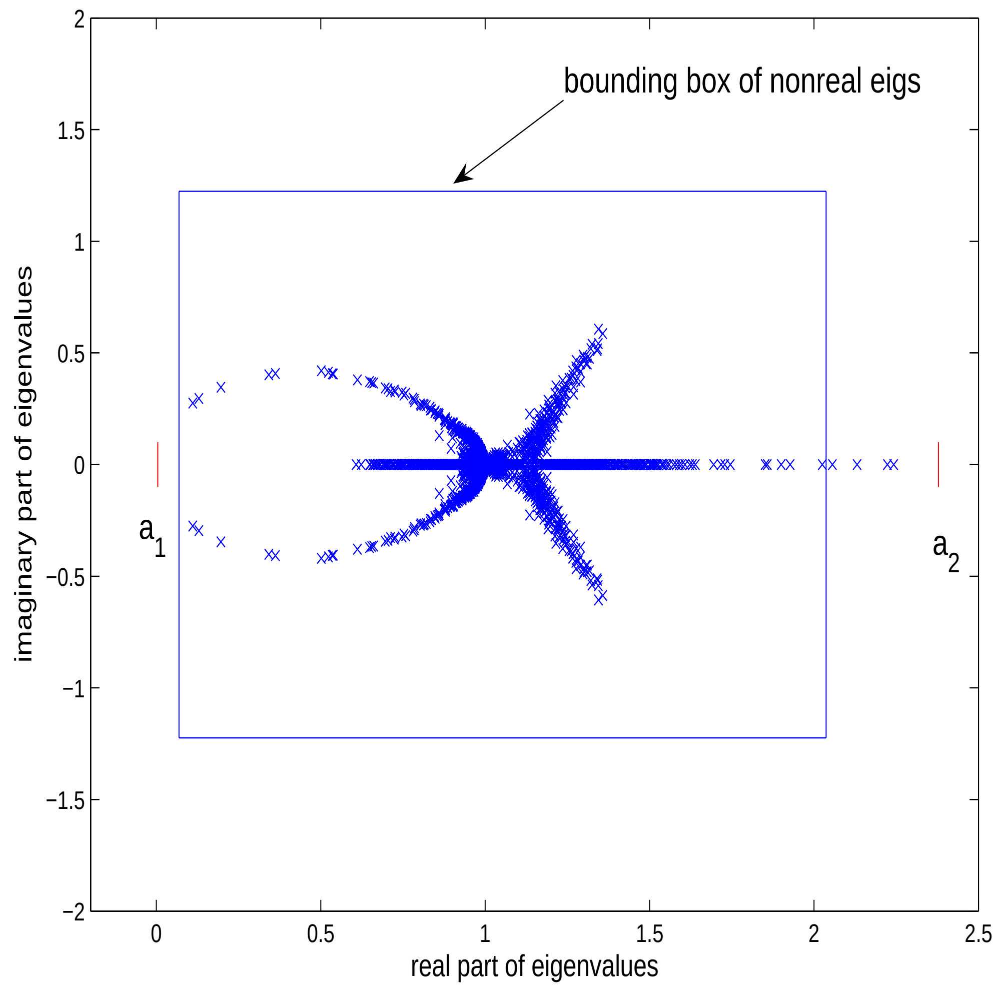
$$\frac{1}{2}(\lambda_{\min}(A) + \lambda_{\min}(G)) \leq \quad \Re(\lambda) \quad \leq \frac{1}{2}(\lambda_{\max}(A) + \lambda_{\max}(G))$$

$$|\Im(\lambda)| \quad \leq \sigma_{\max}((I - A)B^T(BB^T)^{-\frac{1}{2}}).$$

$\star$ If $\Im(\lambda) = 0$ then

$$\min\{\lambda_{\min}(A), \lambda_{\min}(G)\} \leq \lambda \leq \max\{\lambda_{\max}(A), \lambda_{\max}(G)\}$$

Spectral bounds



$S = \texttt{cholinc}(BB^T, 10^{-2})$

A class of indefinite preconditioners

$$\mathcal{P} = \begin{bmatrix} I & O \\ B\widehat{A}^{-1} & I \end{bmatrix} \begin{bmatrix} \widehat{A} & O \\ O & -S \end{bmatrix} \begin{bmatrix} I & \widehat{A}^{-1}B^T \\ O & I \end{bmatrix}$$

where $\widehat{A} \approx A$, $\quad S \approx B\widehat{A}^{-1}B^T$

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Two alternatives:

$\star$ Use $\mathcal{P}$ with nonsymmetric solver

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where $\widehat{A} \approx A$, $\quad S \approx B\widehat{A}^{-1}B^T$

Two alternatives:

⋆ Use $\mathcal{P}$ with nonsymmetric solver

⋆ Enforce constraints on $\widehat{A}$, S to derive H-symmetric CG:

$\mathcal{P}^{-1}\mathcal{M}$ is H-symmetric with

$$H = \begin{bmatrix} \widehat{A} - A & O \\ O & B\widehat{A}^{-1}B^T - S \end{bmatrix} > 0$$

Class of indefinite preconditioners. Stokes problem.

Preconditioner: $\widehat{A}$: $\texttt{cholinc}(A, 10^{-2})$ $\quad S$: $\texttt{cholinc}(B\widehat{A}^{-1}B^T, 10^{-2})$

Preconditioner forcing H-symmetry: $\widetilde{A} = \widehat{A} + \tau I$, s.t. $\lambda_{\min}(\widetilde{A} - A) \approx 2.74$

$\widehat{S}$: from $\texttt{cholinc}(B\widetilde{A}^{-1}B, 10^{-2})$ scaled s.t. $\lambda_{\min}(B\widetilde{A}^{-1}B^T - \widehat{S}) \approx 0.07$

Class of indefinite preconditioners. Stokes problem.

Preconditioner: $\widehat{A}$: $\texttt{cholinc}(A, 10^{-2})$ S : $\texttt{cholinc}(B\widehat{A}^{-1}B^T, 10^{-2})$

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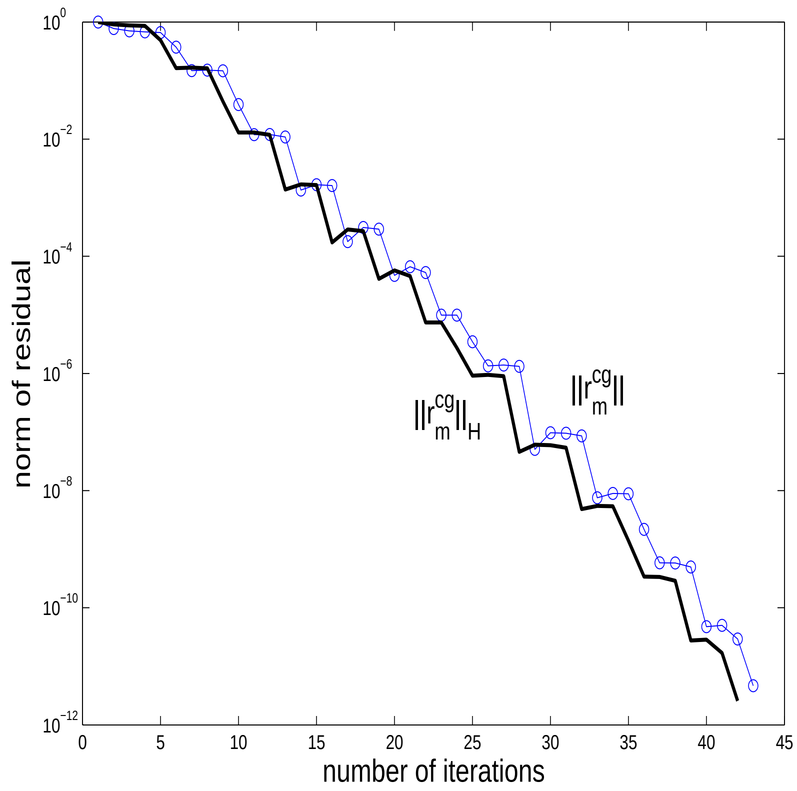

Class of indefinite preconditioners. Stokes problem.

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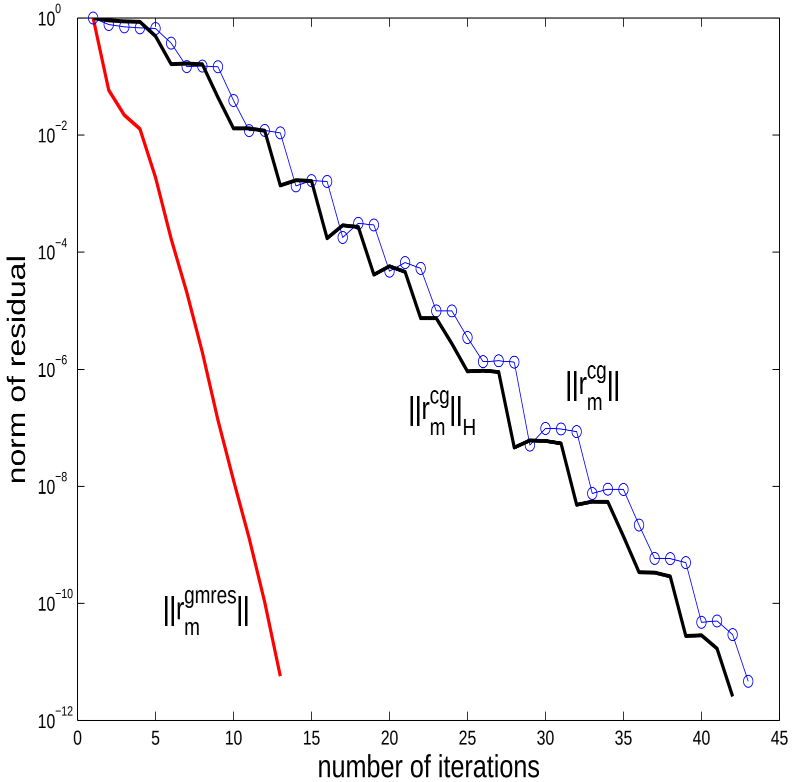
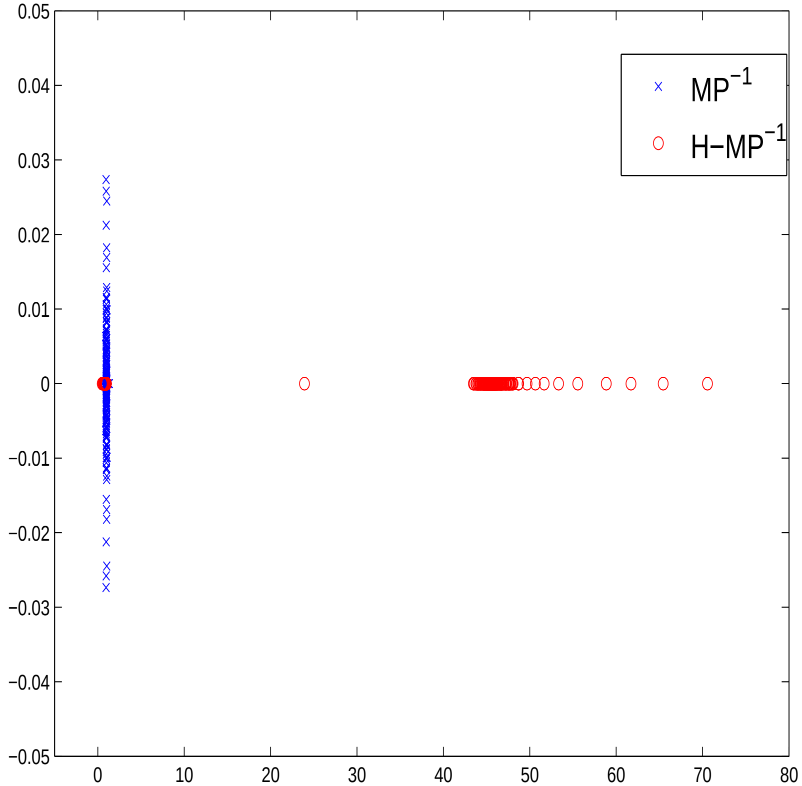
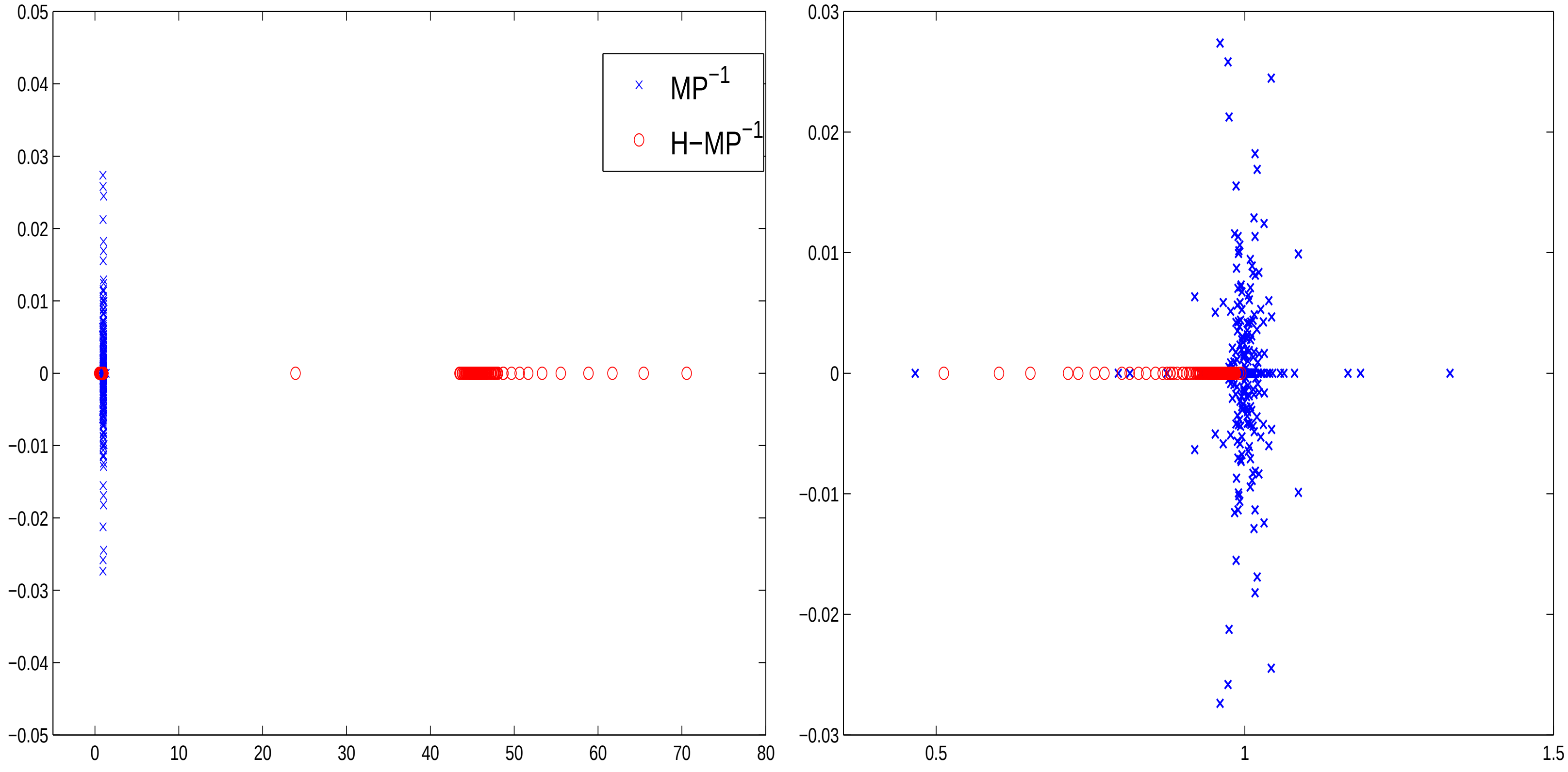


Class of indefinite preconditioners. Spectra.



Class of indefinite preconditioners. Spectra.



Conclusions

- Use H-symmetric solvers when problem knowledge (with moderation)
- More confidence with nonsymmetric solvers
- More theoretical work needed for nonsymmetric preconditioners (and solvers)