Size

Tom Leinster

Glasgow/EPSRC

Plan

1. Cardinality-like invariants

2. The cardinality of a category

3. The cardinality of an enriched category

4. A more conceptual approach

bring together
many notions of
size in mathematics

1. Cardinality-like invariants

Sets	have	cardinality
Topological spaces	have	Euler characteristic
O-minimal sets	have	Euler characteristic (van den Dries)
Measurable sets	have	measure
Polyconvex sets	have	intrinsic volumes (Hadwiger)
Vector spaces	have	dimension
Probability spaces	have	exponential of entropy

Typical properties:

- $|A \times B| = |A| \times |B|$
- $|A \cup B| = |A| + |B| - |A \cap B|$

(suitably interpreted)

Structure-forgetting functors tend *not* to preserve ‘size’, e.g. **Vect** $\to$ **Set**.

Free functors *do* tend to preserve size, e.g. **Set** $\to$ **Vect**.

2. The cardinality of a category

Plan: Define and explore an invariant, the cardinality, of finite categories.

Possible motivation: a categorical view of combinatorics

Inclusion-exclusion formula:

$$|X_1 \cup X_2| = |X_1| + |X_2| - |X_1 \cap X_2|.$$

Categorical version

Let $\mathbf{L} = \begin{pmatrix} 0 & \longrightarrow & 1 \\ \downarrow & & \\ 2 & & \end{pmatrix}$.

For 'good' functors $X : \mathbf{L} \to \mathbf{FinSet}$,

$$|\varinjlim X| = |X_1| + |X_2| - |X_0|.$$

Free action by a group:

$$|X/G| = \frac{1}{o(G)}|X|.$$

Categorical version

Let $\mathbf{G}$ be the group G viewed as a category with one object, $\star$.

For 'good' functors $X : \mathbf{G} \to \mathbf{FinSet}$,

$$|\varinjlim X| = \frac{1}{o(G)}|X(\star)|.$$

Possible motivation: a categorical view of combinatorics

Is there a common generalization?

For each finite category $\mathbf{A}$, we seek weights $w_a \in \mathbb{Q}$ ($a \in \mathbf{A}$) such that for all 'good' functors $X : \mathbf{A} \to \mathbf{FinSet}$,

$$|\varinjlim X| = \sum_{a \in \mathbf{A}} w_a |X(a)|.$$

If such weights exist, $\sum_a w_a$ might be a useful measure of the category $\mathbf{A}$.

This works out as follows. . .

The cardinality of a category: definition

Let $\mathbf{A}$ be a finite category.

A weighting on $\mathbf{A}$ is a family $(w_a)_{a\in\mathbf{A}}$ of rationals such that

$$\forall a \in \mathbf{A}, \qquad \sum_b |\mathrm{Hom}(a,b)| w_b = 1.$$

Coweightings $(w^a)_{a\in\mathbf{A}}$ are defined dually:

$$\forall b \in \mathbf{A}, \qquad \sum_a |\mathrm{Hom}(a,b)| w^a = 1.$$

Easy Lemma

If $w_\bullet$ is a weighting and $w^\bullet$ is a coweighting then $\sum_a w_a = \sum_a w^a$.

Assuming that there exist a weighting $w_\bullet$ and a coweighting $w^\bullet$ on $\mathbf{A}$, define the cardinality of $\mathbf{A}$ as

$$|\mathbf{A}| = \sum_{a\in\mathbf{A}} w_a = \sum_{a\in\mathbf{A}} w^a \in \mathbb{Q}.$$

This is independent of the choice of weighting and coweighting.

The cardinality of a category: elementary examples

Example

Let $\mathbf{A} = \begin{pmatrix} 0 & \longrightarrow & 1 \\ \downarrow & & \\ 2 & & \end{pmatrix}$. The unique weighting $w_\bullet$ is $w_0 = -1$, $w_1 = w_2 = 1$. So

$$|\mathbf{A}| = -1 + 1 + 1 = 1.$$

Example

Let $\mathbf{A} = (0 \rightrightarrows 1)$. The unique weighting $w_\bullet$ is $w_0 = -1$, $w_1 = 1$. So

$$|\mathbf{A}| = 0 \ (= \chi(S^1)).$$

Comparison with cardinality of a set

Given a set S, let $\mathbf{S}$ be the discrete category on S: objects are elements of S, and there are no morphisms except identities.

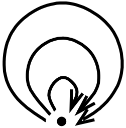
Theorem

Let S be a finite set. Then

$$|\mathbf{S}| = |S|.$$

Comparison with order of a monoid

Given a monoid G, let $\mathbf{G}$ be the one-object category whose morphisms are the elements of G.



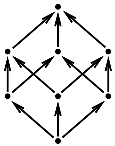
Write $\mathrm{o}(G) = (\text{order of } G)$.

Theorem

Let G be a finite monoid. Then

$$|\mathbf{G}| = \frac{1}{\mathrm{o}(G)}.$$

Comparison with Euler characteristic of a poset



Given a poset P, let $\mathbf{P}$ be the category whose objects are the elements of P and with

$$\mathrm{Hom}(a, b) = \begin{cases} 1 & \text{if } a \leq b \\ 0 & \text{otherwise} \end{cases}$$

Write

$$\chi(P) = \sum_{n=0}^{\infty} (-1)^n |\{\text{chains } a_0 < a_1 < \cdots < a_n \text{ in } P\}|.$$

Theorem

Let P be a finite poset. Then

$$|\mathbf{P}| = \chi(P).$$

Comparison with Euler characteristic of a graph

Given a directed graph G, let $F(G)$ be the category whose objects are the vertices of G and whose morphisms are the directed paths.

Write

$$\chi(G) = (\text{no. vertices of } G) - (\text{no. edges of } G).$$

Theorem

Let G be a finite, circuit-free graph. Then

$$|F(G)| = \chi(G).$$

Comparison with Euler characteristic of a topological space

Given a category $\mathbf{A}$, let $B\mathbf{A}$ be its classifying space (the geometric realization of its nerve).

E.g.: If $\mathbf{A} = (0 \rightrightarrows 1)$ then $B\mathbf{A} = S^1$.

Theorem

Let $\mathbf{A}$ be a finite category containing no endomorphisms or isomorphisms except identities. Then

$$|\mathbf{A}| = \chi(B\mathbf{A}) \in \mathbb{Z}.$$

Comparision with Euler characteristic of an orbifold

Given a triangulated orbifold X, can build a category $\mathbf{S}(X)$ (Moerdijk and Pronk).

E.g.: If X is a manifold then $\mathbf{S}(X)$ is the poset of simplices in the triangulation.

Theorem (with Ieke Moerdijk)

Let X be a finitely-triangulated orbifold. Then

$$|\mathbf{S}(X)| = \chi(X) \in \mathbb{Q}.$$

Properties of the cardinality of categories

Invariance: $\mathbf{A} \simeq \mathbf{B} \Rightarrow |\mathbf{A}| = |\mathbf{B}|$

Products: $|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}| \times |\mathbf{B}|$

Coproducts: $|\mathbf{A} \amalg \mathbf{B}| = |\mathbf{A}| + |\mathbf{B}|$

Fibrations: There is a sensible formula for the cardinality of a fibred category.

Plan

1. Cardinality-like invariants

2. The cardinality of a category

3. The cardinality of an enriched category

4. A more conceptual approach

3. The cardinality of an enriched category

When we defined the cardinality of a category, all we used about the sets $\mathrm{Hom}(a, b)$ was their cardinalities.

We didn't even use the fact that they were sets.

Idea: We can replace the *sets* $\mathrm{Hom}(a, b)$ by *any* type of object for which there is a good notion of cardinality.

Background: enriched categories

Approximate definitions:

A monoidal category is a category $\mathscr{V}$ equipped with a binary product $\otimes$, associative and unital up to isomorphism.

E.g.: $\mathscr{V} = \mathbf{Set}$ with $\otimes = \times$; or $\mathscr{V} = \mathbf{Vect}$ with usual $\otimes$.

A category enriched in $\mathscr{V}$ consists of:

- a class $\mathrm{ob}\,\mathbf{A}$ of objects
- for each $a, b \in \mathrm{ob}\,\mathbf{A}$, an object $\mathrm{Hom}(a, b) \in \mathscr{V}$
- for each $a, b, c \in \mathrm{ob}\,\mathbf{A}$, a morphism

$$\mathrm{Hom}(a, b) \otimes \mathrm{Hom}(b, c) \to \mathrm{Hom}(a, c)$$

  in $\mathscr{V}$ ('composition'),

etc.

E.g.: A category enriched in $(\mathbf{Set}, \times)$ is an ordinary category.

A category enriched in $(\mathbf{Vect}, \otimes)$ is a 'linear category'.

The cardinality of an enriched category: definition

Suppose that we have a monoidal category $\mathscr{V}$
and a 'cardinality' $|X|$ for each object $X \in \mathscr{V}$.

Using exactly the same words as before, we obtain a definition of the cardinality $|\mathbf{A}|$ of a finite category $\mathbf{A}$ enriched in $\mathscr{V}$.

Example

Take $\mathscr{V} = \mathbf{Vect}$, usual $\otimes$, and $|X| = \dim(X)$. Then we obtain a definition of the cardinality of a linear category with finitely many objects and finite-dimensional hom-spaces $\mathrm{Hom}(a, b)$.

Comparison with Euler characteristic of an algebra

Given a k-algebra A, let $\mathbf{A}$ be the category enriched in $\mathbf{Vect}_k$ whose objects are the projective indecomposable A-modules, and with $\mathrm{Hom}(M, N) = \mathrm{Hom}_A(M, N)$.

For a graded algebra A, write

$$\chi(A) = \sum_{i=0}^{\infty} (-1)^i \dim(\mathrm{Ext}^i_A(A_0, A_0)).$$

Theorem (with Catharina Stroppel)

Let A be a Koszul algebra, of finite dimension and finite global dimension, over an algebraically closed field. Then

$$|\mathbf{A}| = \chi(A).$$

The cardinality of an n-category

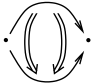
A (small) 0-category is a set.

A (small, strict) n-category is a category enriched in (($n-1$)-categories).

From the notion of the cardinality of a finite set, we obtain inductively a notion of the cardinality of a finite n-category.

Example

For each n, there is an n-category $\mathbf{S}^n$ consisting of two parallel n-morphisms.

(E.g. $\mathbf{S}^2 =$  .) Then

$$|\mathbf{S}^n| = 1 + (-1)^n = \chi(S^n).$$

Background: metric spaces as enriched categories

Observation (Lawvere): A metric space is an enriched category.

Let $\mathscr{V}$ be the poset $([0, \infty], \geq)$ viewed as a category:

- objects of $\mathscr{V}$ are numbers $x \in [0, \infty]$
- there is one morphism $x \to y$ if $x \geq y$, and there are none otherwise.

$\mathscr{V}$ is a monoidal category with $\otimes = +$.

Any metric space A is naturally a category enriched in $\mathscr{V}$:

- objects are points
- $\text{Hom}(a, b) = d(a, b) \in \mathscr{V}$
- composition becomes the triangle inequality $d(a, b) + d(b, c) \geq d(a, c)$.

The cardinality of a finite metric space

Define the cardinality of $x \in [0, \infty]$ to be e^{-2x}. (There is a good reason.)

We obtain a definition of the cardinality of a finite metric space.

Explicitly: Let A be a finite metric space. Solve the system of equations

$$\forall a \in A, \qquad \sum_{b \in A} e^{-2d(a,b)} w_b = 1$$

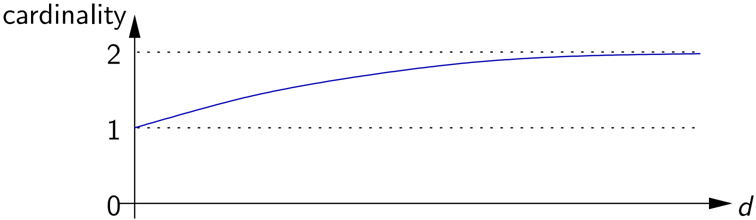
in real numbers w_b ($b \in A$). Then $|A| = \sum_a w_a$.

The cardinality of a finite metric space: examples

E.g.: $|\emptyset| = 0$.

E.g.: $|\bullet| = 1$.

E.g.: $|\bullet \overset{\leftarrow\ d\ \rightarrow}{\qquad} \bullet| = 1 + \tanh d$:



E.g.: If A has n points with $d(a, b) = \infty$ for all $a \neq b$ then $|A| = n$.

Comparison with entropy of a probability distribution

There is a notion of the entropy $H_A(\mathbf{p})$ of a probability distribution $\mathbf{p}$ on a finite metric space.

The maximum entropy problem: fix a background space A.
Which probability distribution $\mathbf{p}$ maximizes the entropy $H_A(\mathbf{p})$?

Theorem

Let A be a finite, 'well-separated' metric space. Then

$$|A| = \exp(\sup_{\mathbf{p}} H_A(\mathbf{p}))$$

where the supremum is over all probability distributions $\mathbf{p}$ on A.

So cardinality is exp(maximum entropy).

The cardinality of a compact metric space

Idea: Given a compact metric space A, choose a sequence

$$A_0 \subseteq A_1 \subseteq \cdots$$

of finite subsets of A, with $\bigcup_i A_i$ dense in A. *Try* to define

$$|A| = \lim_{i \to \infty} |A_i|.$$

Sometimes this works! E.g.

$$|[0, s] \times [0, t]| \ = \ 1 + (s + t) + st$$

Euler char

$\frac{1}{2}\times$perimeter

area

degree is dimension

(using the appropriate product metric: 'd_1').

4. A more conceptual approach

Problem: Cardinality is sometimes undefined.

'Motivic' solution: Let cardinality live where it wants to live!

Strategy: (For simplicity, just describe non-enriched case.)

Given a category $\mathbf{A}$, let $R(\mathbf{A})$ be the semiring generated by elements w_a and w^a ($a \in \mathbf{A}$) subject to the usual equations:

$$\forall a, \quad \sum_b |\mathrm{Hom}(a,b)| w_b = 1; \qquad \forall b, \quad \sum_a |\mathrm{Hom}(a,b)| w^a = 1$$

Define $\|\mathbf{A}\| = \sum_a w_a = \sum_a w^a \in R(\mathbf{A})$.

Easy Lemma: $\|\mathbf{A}\| \in R(\mathbf{A})$ is absolute, that is, for all semirings S and homomorphisms $f, g : R(\mathbf{A}) \to S$, we have $f(\|\mathbf{A}\|) = g(\|\mathbf{A}\|)$.

So *if* there exists $f : R(\mathbf{A}) \to \mathbb{Q}$ then $f(\|\mathbf{A}\|)$ is independent of choice of f. This rational number is the old cardinality $|\mathbf{A}|$.

Summary

There is a canonical notion of
the cardinality of a finite category

It links together, by comparison theorems, many notions of ‘size’ in mathematics. E.g.: size of sets, groups, graphs, posets, topological spaces, manifolds and orbifolds. More generally:

There is a canonical notion of
the cardinality of a finite enriched category

It links together several more notions of ‘size’: e.g. size of associative algebras and probability distributions. It also provides some new invariants: e.g. the cardinality of a metric space. But so far, it is less explored.

Notes and references

These slides are available at www.maths.gla.ac.uk/∼tl

Cardinality-like invariants: I have learned most about the general idea of 'size' from

John Baez, The mysteries of counting: Euler characteristic versus homotopy cardinality, http://math.ucr.edu/home/baez/counting

and

Daniel A. Klain, Gian-Carlo Rota, *Introduction to Geometric Probability*, Lezioni Lincee, Cambridge University Press, 1997

and two papers of Schanuel, who was the first to really push the idea that Euler characteristic is to topological spaces as cardinality is to sets:

Stephen H. Schanuel, Negative sets have Euler characteristic and dimension, *Category Theory (Como, 1990)*, 379–385, Lecture Notes in Mathematics 1488, Springer, 1991

Notes and references

and

Stephen H. Schanuel, What is the length of a potato? An introduction to geometric measure theory, in *Categories in Continuum Physics*, Lecture Notes in Mathematics 1174, Springer, 1986.

Klain and Rota's book also contains a good account of Hadwiger's theorem on the intrinsic volumes of polyconvex sets. The Euler characteristic of o-minimal sets was defined in

Lou van den Dries, *Tame Topology and O-minimal Structures*, London Mathematical Society Lecture Note Series 248, Cambridge University Press, 1998.

Notes and references

The cardinality of a category: The main paper on this is

> Tom Leinster, The Euler characteristic of a category, *Documenta Mathematica* 13 (2008), 21–49,
> www.math.uni–bielefeld.de/documenta/vol–13/02.html

The 'good' functors mentioned in the slides are the sums of representables. When **A** is Cauchy-complete, this is equivalent to being flat with respect to finite connected limits.

A different approach, perhaps more appealing to homotopy theorists, is in

> Clemens Berger, Tom Leinster, The Euler characteristic of a category as the sum of a divergent series, *Homology, Homotopy and Applications* 10(1) (2008), 41–51,
> http://intlpress.com/HHA/v10/n1/a3

What I called the cardinality of a category here is called its Euler characteristic in these papers.

Notes and references

The cardinality of an enriched category: Many introductions to category theory contain an account of enriched categories. Categories enriched in $\mathscr{V}$ are sometimes referred to as '$\mathscr{V}$-categories'.

The discovery that metric spaces are enriched categories was published by Lawvere in 1973; that classic paper has been reprinted as

> F. William Lawvere, Metric spaces, generalized logic, and closed categories, Reprints in Theory and Applications of Categories 1 (2002), 1–37.

The work on metric spaces has yet to be written up properly. The existing sources are

> Tom Leinster, The cardinality of a metric space, post at *The n-Category Café*, 9 February 2008, http://golem.ph.utexas.edu/category/2008/02/metric_spaces.html

(which is detailed but contains some mistakes) and

Notes and references

Tom Leinster, The cardinality of a metric space, talk at CT08, Calais, www.maths.gla.ac.uk/∼tl/calais

Cardinality is not quite a new invariant of metric spaces. It was previously discovered (though not much developed) in a completely different context:

Andrew Solow, Stephen Polasky, Measuring biological diversity, *Environmental and Ecological Statistics* 1 (1994), 95–107.

The relationship between cardinality of a metric space, entropy of a probability space, and diversity of an ecosystem is discussed here:

Tom Leinster, Entropy, diversity and cardinality (part 2), post at *The n-Category Café*, 7 November 2008, http://golem.ph.utexas.edu/category/2008/11/entropy_diversity_and_cardinal_1.html

The result on cardinality and the exponential of maximal entropy has not been published. 'Well-separated' means that the distance between distinct points is $> \frac{1}{2}\log(N-1)$, where N is the number of points of A.