The Continuous Discontinuous Finite Element Method

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Model Problem

We study the advection-diffusion equation

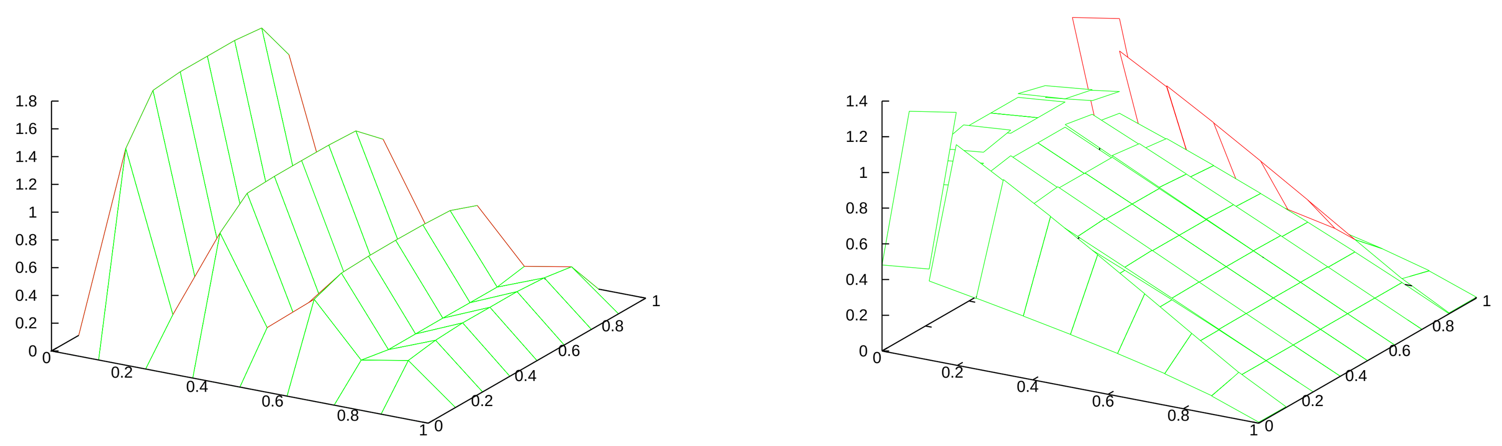
$$-\varepsilon\Delta u + b\cdot\nabla u = f \qquad \text{in } \Omega\subset\mathbb{R}^d \tag{1a}$$

$$u = 0 \qquad \text{on } \partial\Omega \tag{1b}$$

in a polygonal domain Ω with $\varepsilon\in\mathbb{R}_+$, $b\in W^\infty(\mathrm{div},\Omega)$ and $f\in L^2(\Omega)$ where we define $W^\infty(\mathrm{div},\Omega) := \{v\in[L^\infty(\Omega)]^d;\ (\nabla\cdot v)\in L^\infty(\Omega)\}$. It is a standard result that (1) has a unique solution in $H^1_0(\Omega)$ provided that $-\frac{1}{2}\nabla\cdot b\geq 0$. The first term in (1a) represents diffusion, and the second advection.

cG and dG Methods

The standard continuous Galerkin (cG) finite element method applied to (1) is simple to implement but has poor performance as $\varepsilon\to 0$ and non-physical oscillations are apparent in the solution. Figure 1(a) shows the cG solution to (1) with $\varepsilon = 0.01$, $b = (-1,0)^\top$ and $f = 1$ using bilinear elements on a 8×8 grid with the oscillations clearly apparent.



(a) Unstable cG solution (b) dG solution

Figure 1: The cG and dG solutions

It is possible to introduce modifications to the basic cG method to stabilize the problem and for a review of techniques readers should consult e.g. [4].

An alternative, naturally stable family of methods for the solution of (1) are the IP discontinuous Galerkin (dG) methods. For (1) the IP methods are stable given certain conditions on the advection term and penalty terms in the method [1]. The drawback of such methods is the marked increase in degrees of freedom, thus augmenting considerably the computational cost. Figure 1(b) shows the dG solution for (1). Unlike the cG solution in Figure 1(a), under and over shooting are limited to the region in close proximity to the true solution's layers. Here we have used the nonsymmetric IP method and the penalty parameter $\sigma = 10$. Note that although the numerical method allows jumps the numerical solution is almost continuous away from layers. The dG solution with linear elements shown here has 256 degrees of freedom compared to 81 for the cG method. The near continuity on much of the mesh and the increase in degrees of freedom leads us to propose a joint continuous discontinuous Galerkin (cdG) method. We also look to more fully understand the role of the extra degrees of freedom in the dG method. Our goal is to achieve stability while minimizing the degrees of freedom.

cdG Methods

It is the exponential boundary layer that causes the cG solution to have non-physical oscillations. We therefore propose to use the dG method on a region of Ω that covers any boundary layers and the cG method away from layers. The advantages of such a method are a significant reduction in the degrees of freedom required compared with a pure dG method, yet still maintaining stability. This method was first proposed in [2] and is an extension of the method proposed in [3].

We call the region where the cG and dG methods are applied $\Omega_{\mathscr{D}}$ and $\Omega_{\mathscr{C}}$ respectively, and likewise the triangulations on these regions $\mathscr{T}_{\mathscr{D}}$ and $\mathscr{T}_{\mathscr{C}}$. We decompose Ω by considering the region where the true solution u is close to the solution of the hyperbolic reduced problem obtained by imposing $\varepsilon = 0$. In this region Ω_0 it is assumed that the difference $u_{\varepsilon\Delta}$ between the true solution and the reduced solution satisfies the bounds

$$\forall\varepsilon\in(0,1]: \|u_{\varepsilon\Delta}\|_{L^2(\Omega_0)} < C_0\varepsilon \quad\text{and}\quad |u_{\varepsilon\Delta}|_{H^1(\Omega_0)} < C_0\varepsilon^{1/2}. \tag{2}$$

See [4], Chapter III.1 for justification of these assumptions. Based on this condition we construct our decomposition.

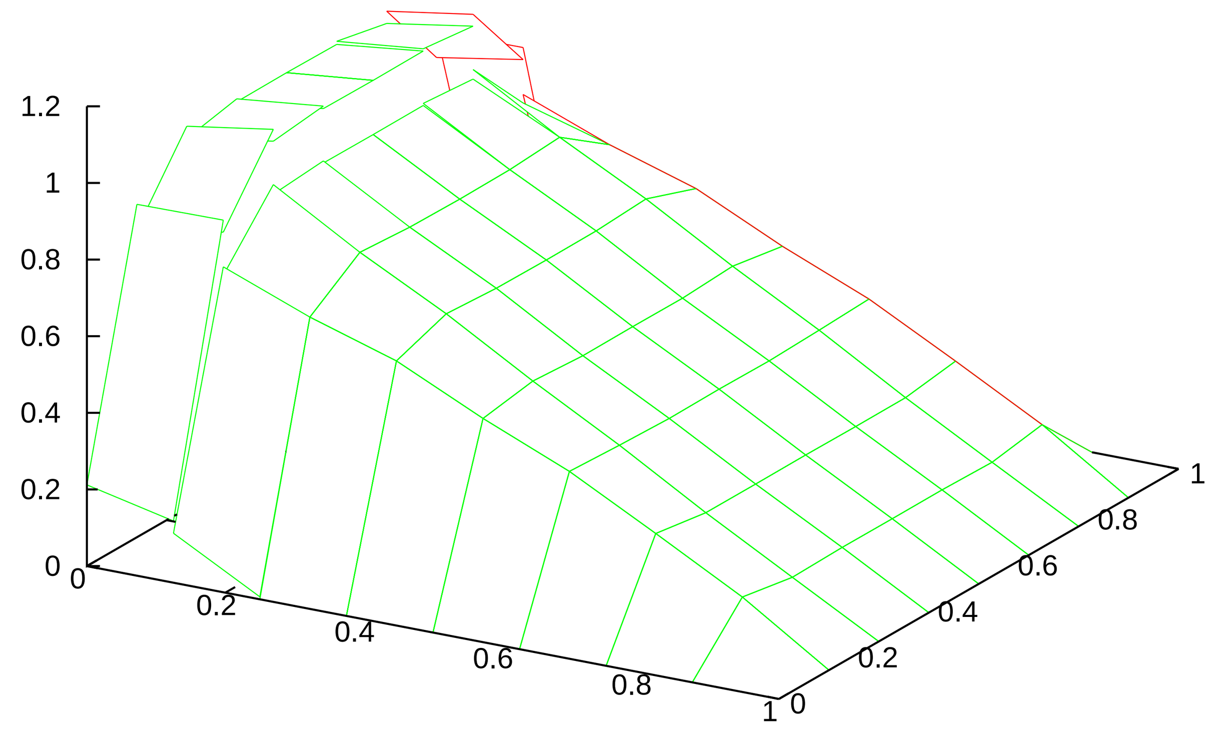


Figure 2: The cdG Solution

We have proved that the cdG method is stable on the region $\Omega_{\mathscr{D}}$ using and extending the method of [1]. In the $\Omega_{\mathscr{C}}$ region it is more difficult to derive an a priori error estimate. We have so far been unable to find a rigorous epsilon-independent estimate on this region, but have a partial epsilon-independent result that localizes the difficulties to the interface between the $\mathscr{T}_{\mathscr{C}}$ and $\mathscr{T}_{\mathscr{D}}$ elements. Numerical evidence suggests epsilon-independent stability. Figure 2 shows the cdG solution to the advection-diffusion equation presented in the previous section, with $\Omega_{\mathscr{C}}$ being the region where $x\geq 0.2$. The behaviour of the cdG solution in both regions is close to the behaviour of the dG solution. In addition the boundary conditions are strongly enforced on the boundary in the $\Omega_{\mathscr{C}}$ region. The number of degrees of freedom in this example is 111, considerably fewer than for the dG method.

Our numerical experiments lead us to believe the method will be stable in both regions. We hope to be able to improve our proof on the $\Omega_{\mathscr{C}}$ region. Further work will also include details for practical implementation of the method. For example with more complex (non-constant) advection coefficients or domains it may not be straightforward a priori to perform the decomposition, and so some algorithm will be required to implement the method.

References

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