A Bayesian approach to an elliptic inverse problem

M. DASHTI, S. HARRIS, A. STUART

Mathematics Institute, University of Warwick

1. The problem

- Consider (Darcy's law + incompressibilty condition):

$$\nabla \cdot \Big(\exp(u)\nabla p\Big) = 0, \quad x \in D \subset \mathbb{R}^d, \quad \text{d=2,3} \qquad (1)$$
$$p = \phi, \quad x \in \partial D$$

- Find u, Given noisy *observations* of p at a set of points $x_1, \ldots, x_K$ in D:

$$y_k = p(x_k) + \eta_k, \quad k = 1, \ldots, K. \qquad (2)$$

2. Bayesian approach to inverse problems

- Unknown function $u \in X$ (X Banach space),
- Prior $\mu_0(\mathrm{d}u) = \mathbb{P}(\mathrm{d}u)$ on u with $\mu_0(X) = 1$
- $y|u$ noisy observations $y \in Y$

$$y = \mathcal{G}(u) + \eta$$

$\mathcal{G}$: observational operator
η: observational noise distributed according to $\mathcal{N}(0, \Gamma)$

- Posterior $\mu^y(\mathrm{d}u) = \mathbb{P}(\mathrm{d}u|y)$ on u:

$$\frac{\mathrm{d}\mu^y}{\mathrm{d}\mu_0}(u) \propto \mathbb{P}(y|u) \propto \exp(-\Phi(u,y)), \qquad (3)$$

$\Phi : X \times Y \to \mathbb{R}$.

2.1. Prior measure

Let $\{\psi_l\}_{l=1}^\infty$ be a basis for $L^2(D)$. Define random function u as

$$u(x) = \sum_{l=1}^{\infty} l^{-(\frac{s}{d}+\frac{1}{2}-\frac{1}{p})} \left(\tfrac{1}{\kappa}\right)^{\frac{1}{p}} \xi_l\, \psi_l(x). \qquad (4)$$

with $1 \le p < \infty$, $s > 0$, and $\kappa > 0$ fixed and $\{\xi_l\}_{l=1}^\infty$ i.i.d real-valued random variables with probability distribution function

$$\pi_\xi(x) = c_p \exp(-|x|^p)$$

- When $\{\psi_l\}_{l=1}^\infty$ is a Fourier basis, $p = 2$ and $\kappa = \frac{1}{2}$, u is distributed according to the Gaussian measure $\mathcal{N}(0, (-\Delta)^{-s})$ with Δ the Laplacian operator.
In this case $\|u\|_{H^t} < \infty$ a.s. for $t < s - d/2$.
- Let $D = \mathbb{T}^d$. When $\{\psi_l\}_{l=1}^\infty$ in (4) is an r-regular wavelet basis for $L^2(\mathbb{T}^d)$, then u is distributed according to a Besov (κ, B^s_{pp}) measure (formally $\mu_0(\mathrm{d}u) \propto \exp(-\kappa\|u\|^p_{B^s_{pp}})$).
In this case $\|u\|_{B^t_{pp}} < \infty$ a.s. for $t < s - d/p$.

2.2. Wellposedness of the posterior measure

Assumption 1. Function $\Phi : X \times Y \to \mathbb{R}$ satisfies

(i) There is an $\alpha_1 > 0$ and for every $r > 0$, an $M \in \mathbb{R}$, such that for all $u \in X$ and for all $y \in Y$ such that $||y||_Y < r$

$$\Phi(u,y) \ge M - \alpha_1 ||u||_X^p.$$

(ii) For every $r > 0$ there exists $K = K(r) > 0$ such that for all $u \in B^t_{pp}$, $y \in Y$ with $\max\{||u||_X, ||y||_Y\} < r$

$$\Phi(u,y) \le K.$$

(iii) For any $r > 0$ an $L = L(r) > 0$ exists such that $u_1, u_2 \in B^t_{pp}$ and $u \in Y$ with $\max\{||u_1||_X, ||u_2||_X, ||y||_Y\} < r$

$$|\Phi(u_1,y) - \Phi(u_2,y)| \le L||u_1 - u_2||_X.$$

(iv) There is an $\alpha_2 > 0$ and for every $r > 0$ a $C \in \mathbb{R}$ such that for all $y_1, y_2 \in \mathbb{R}$ with $\max\{||y_1||_Y, ||y_2||_Y\} < r$ and for every $u \in X$

$$|\Phi(u,y_1) - \Phi(u,y_2)| \le \exp(\alpha_2 ||u||_X^p + C)||y_1 - y_2||.$$

Assumption 2. The prior measure μ_0 is either

a Gaussian H^s measure with s large enough so that $H^t \subset X$ for some $t < s - \frac{d}{2}$, or

a Besov (κ, B^s_{pp}) measure with s large enough so that $B^t_{pp} \subset X$ for some $t < s - \frac{d}{p}$.

Theorem 1. *Let Assumption 1.(i)–(iii) and Assumption 2 with $\kappa > \alpha_1$ hold. Then μ^y given by (3) is a well-defined probability measure.*

One can also show continuity of the posterior in the Hellinger metric with respect to the data y. The Hellinger metric is defined as follows:

$$d_{\text{Hell}}(\mu, \mu') = \sqrt{\frac{1}{2}\int \left(\sqrt{\frac{\mathrm{d}\mu}{\mathrm{d}\nu}} - \sqrt{\frac{\mathrm{d}\mu'}{\mathrm{d}\nu}}\right)^2 \mathrm{d}\nu}.$$

Theorem 2. *Let Assumption 1 and Assumption 2 with $\kappa > 2\alpha_2$ hold. Then*

$$d_{\text{Hell}}(\mu^y, \mu^{y'}) \le C\,|y - y'|$$

where $C = C(r)$ with $\max\{|y|, |y'|\} \le r$.

2.3. Approximation of the posterior

Let Φ^N be an approximation of Φ.
Define $\mu^{y,N}$ by

$$\frac{\mathrm{d}\mu^{y,N}}{\mathrm{d}\mu_0}(u) = \frac{1}{Z^N(y)} \exp\big(-\Phi^N(u)\big) \qquad (5)$$

where

$$Z^N(y) = \int_X \exp\big(-\Phi^N(u)\big)\mathrm{d}\mu_0(u).$$

Theorem 3. *Suppose that Φ and Φ^N satisfy Assumption 1.(i)–(iii) uniformly in N. Let Assumption 2 hold. If*

$$|\Phi(u) - \Phi^N(u)| \le C\psi(N)$$

where $\psi(N) \to 0$ as $N \to \infty$, then there exists a constant independent of N such that

$$d_{Hell}(\mu^y, \mu^{y,N}) \le C\psi(N).$$

3. Application to the elliptic inverse problem

- In (1) for any $u \in L^\infty(D)$ we assume that

$$\lambda(u) = \operatorname{ess\,inf}_{x \in D} \mathrm{e}^{u(x)} > 0$$
$$\Lambda(u) = \operatorname{ess\,sup}_{x \in D} \mathrm{e}^{u(x)} < \infty.$$

- We do not assume that the upper and lower bounds on λ/Λ hold uniformly across the probability space.

Observations: given as in (2), and we assume that:

the noise is Gaussian and
$\{\eta_k\}$ is an i.i.d sequence with $\eta_1 \sim \mathcal{N}(0, \gamma^2 I)$.

Concatenating the data, we have

$$y = \mathcal{G}(u) + \eta$$

where

$$y = (y_1, \cdots, y_K)^T;$$
$$\eta = (\eta_1, \ldots, \eta_K)^T \sim \mathcal{N}(0, \gamma^2 I)$$
$$\mathcal{G}(u) = (p(x_1), \cdots, p(x_K))^T, \quad \text{the observation operator.}$$

Estimates:

- If $u \in L^\infty(D)$ there exists $C = C(D, \|\phi\|_{L^\infty(\partial D)})$ such that

$$|\mathcal{G}(u)| \le C\, \mathrm{e}^{\|u\|_{L^\infty}}.$$

- If $u_1, u_2 \in C^t(D)$ for some $t > 0$ then there exists $C = C(D,t)$ such that

$$|\mathcal{G}(u_1) - \mathcal{G}(u_2)|$$
$$\le C\, \exp\left(\max\{\|u_1\|_{C^t}, \|u_2\|_{C^t}\}\right) \|u_1 - u_2\|_{L^\infty}.$$

Therefore by Theorems 1 and 2, we have:

Wellposedness: If μ_0 satisfies Assumption 2 with $\kappa > 2$ and $s > 2d/p$, then the posterior measure μ^y is absolutely continuous with Radon-Nikodym derivative

$$\frac{\mathrm{d}\mu^y}{\mathrm{d}\mu_0}(u) \propto \exp\Big(-\frac{1}{2\gamma^2}|y - \mathcal{G}(u)|^2\Big)$$

and

$$d_{Hell}(\mu^y, \mu^{y'}) \le C\,|y - y'|$$

for any $y, y' \in \mathbb{R}^K$.

Approximations:

Fourier basis:

Let $\{\psi_l\}_{l \in \mathbb{Z}}$ be the Fourier basis for $L^2(\mathbb{T}^d)$,

$$P^N u = u^N = \sum_{l=1}^{N} u_l\, \psi_l.$$

and $\mathcal{G}^N(\cdot) = \mathcal{G}(P^N \cdot)$.

Theorem 4. *If the prior μ_0 is a Gaussian H^s measure with $s > d + t$, then*

$$d_{\text{Hell}}(\mu^y, \mu^{y,N}) \le C\, N^{-t}(\log N)^d.$$

Wavelet basis:

Let $\{\psi_j\}_{j=1}^\infty$ be an r-regular wavelet basis for $L^2(\mathbb{T}^d)$ and define

$$P^N u = u^N(x) = \sum_{l=1}^{N} u_l\, \psi_l(x)$$

and $\mathcal{G}^N(\cdot) = \mathcal{G}(P^N \cdot)$.

Theorem 5. *If the prior μ_0 is a Besov (κ, B^s_{pp}) measure with $s > 2d/p + t$ and $\kappa > 2$, then*

$$d_{\text{Hell}}(\mu^y, \mu^{y,N}) \le C\, N^{-t/d}.$$